GENERALIZED FIXED-POINT ALGEBRAS AND SQUARE-INTEGRABLE REPRESENTATIONS OF TWISTED C^*-DYNAMICAL SYSTEMS

LEONARD HUANG

ABSTRACT. This paper shows that Ralf Meyer's theory of square-integrable group representations of C^*-dynamical systems can be generalized quite naturally to the case of twisted C^*-dynamical systems. An outcome of this is a generalized fixed-point algebra that is Morita-Rieffel equivalent to a closed two-sided ideal of the reduced twisted crossed product corresponding to the twisted C^*-dynamical system. This paper was inspired by Rieffel's work involving an action of a locally compact Hausdorff group on the C^*-algebra $\mathbb{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$. As such an action can only correspond to a projective representation in general (thus producing a twisted C^*-dynamical system), it seemed only natural to study square-integrable representations of such twisted systems by adapting Meyer's existing framework.

1. INTRODUCTION

In [8], Marc Rieffel defines a strongly continuous action α of a locally compact Hausdorff group G on a C^*-algebra A to be **proper** iff there exists a dense $*$-subalgebra A_0 of A satisfying the following properties:

- For each $a, b \in A_0$, the functions

$$\langle a|b \rangle_E := \left\{ \begin{array}{ccc} G & \to & B \\ x & \mapsto & \Delta(x)^{-1} \cdot a \; \alpha_x(b^*) \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{ccc} G & \to & B \\ x & \mapsto & a \; \alpha_x(b^*) \end{array} \right\}$$

are elements of $\mathscr{L}^1(G, A)$.

- If

$$M(A_0) := \{ m \in M(A) \mid m A_0 \subseteq A_0 \text{ and } A_0 m \subseteq A_0 \},$$
$$M(A_0)^\alpha := \{ m \in M(A_0) \mid (\forall x \in G)(\overline{\alpha_x}(m) = m) \},$$

then for each $a, b \in A_0$, there exists an element $\langle a|b \rangle_D \in M(A_0)^\alpha$ (necessarily unique) such that

$$c \langle a|b \rangle_D = \int_G c \; \alpha_x(a^* b) \; \mathrm{d}x$$

for all $c \in A_0$.

Rieffel then does a number of things using this definition. He shows that the algebraic linear span E_0 of elements of the form $\langle a|b \rangle_E$ is a $*$-subalgebra of the $L^1(G, A)$ convolution $*$-algebra, and then he equips A_0 with the structure of a left E_0-module, using the left action of an A-valued function on G on A_0 defined by

$$f \bullet a = \int_G \sqrt{\Delta(x)} \cdot f(x) \; \alpha_x(a) \; \mathrm{d}x,$$

whenever this makes sense. He proceeds to prove that $\langle \cdot | \cdot \rangle_E$ satisfies the axioms of a non-degenerate E_0-valued pre-inner product, and upon equipping E_0 with the norm inherited from $C_r^*(G, A, \alpha)$, he defines $\overline{A_0}$ to be the completion of A_0 with respect to the norm induced by $\langle \cdot | \cdot \rangle_E$. We thus obtain

a left Hilbert module $_E\overline{A_0}$, where E is the completion of E_0 in $C_r^*(G, A, \alpha)$ and happens to be a closed two-sided ideal of it.

The algebraic linear span D_0 of elements of the form $\langle a|b\rangle_D$ acts on $\overline{A_0}$ on the right in the obvious manner, and Rieffel proves that D_0 is a $*$-subalgebra of $M(A_0)^\alpha$ by establishing the compatibility of $\langle\cdot|\cdot\rangle_D$ with $\langle\cdot|\cdot\rangle_E$. Then D_0 can be completed inside $M(A_0)^\alpha$ to a right imprimitivity $(C^*$-)algebra D, which makes $\overline{A_0}$ an imprimitivity (E, D)-bimodule. It is then the case that there exists an anti-homormorphism from D to the C^*-algebra of compact operators on $\overline{A_0}$.

In [9], Rieffel also describes a theory of square-integrable group representations, which we can summarize as follows. Let G be a locally compact Hausdorff group and $(U, \mathcal{H})$ a strongly continuous unitary Hilbert-space representation of G. For $\xi, \eta \in \mathcal{H}$, define a function

$$c_{\xi,\eta} := \begin{Bmatrix} G & \to & \mathbb{C} \\ x & \mapsto & \langle U_x(\xi)|\eta\rangle_\mathcal{H} \end{Bmatrix}.$$

We then say that a vector $\xi \in \mathcal{H}$ is U-**bounded** iff there exists a constant $C_\xi > 0$ such that

$$\|c_{\xi,\eta}\|_2 \leq C_\xi \|\eta\|_\mathcal{H}$$

for all $\eta \in \mathcal{H}$. Clearly, if ξ is U-bounded, then $c_{\xi,\eta} \in \mathscr{L}^2(G, \mathbb{C})$ for all $\eta \in \mathcal{H}$. Conversely, by the Closed Graph Theorem, one can show that if $c_{\xi,\eta} \in \mathscr{L}^2(G, \mathbb{C})$ for all $\eta \in \mathcal{H}$, then ξ is U-bounded. We call $(U, \mathcal{H})$ a **square-integrable group representation** iff the set B_U of U-bounded vectors is dense in $\mathcal{H}$.

The connection between properness and square-integrability is given by Theorem 7.9 of [9]:

Theorem 1. *Let $(U, \mathcal{H})$ be a unitary representation of G, and let α be the corresponding action on $\mathbb{K}(\mathcal{H})$. Then α is integrable iff $(U, \mathcal{H})$ is square-integrable.*

Here, Rieffel declares an action α of G on a C^*-algebra B to be **integrable** iff we have a dense linear span of the set of elements $b \in B_{\geq}$ for which the function $\begin{Bmatrix} G & \to & B \\ x & \mapsto & \alpha_x(b) \end{Bmatrix}$ is strictly integrable (this is also called an α-**integrable** action). He proves that proper actions, in the sense above, are automatically integrable, and so if α is proper, then $(U, \mathcal{H})$ is square-integrable.

In [9], Rieffel tried to replace the class of proper actions by the class of integrable actions, but Ruy Exel showed that the class of integrable actions was simply too general to yield a fixed-point algebra lying inside of $C_r^*(G, B, \beta)$ as a closed two-sided ideal. The class of abelian C^*-dynamical systems already contained a counterexample, which prompted Exel to devise (for this class) a relation, called 'relatively continuous', between integrable elements that guaranteed a fixed-point algebra with the desired property.

In [5], Ralf Meyer generalizes Rieffel's theory by replacing $\mathcal{H}$ by a Hilbert B-module $\mathcal{E}$. As such, the scalar-valued inner product on $\mathcal{H}$ is replaced by the B-valued inner product on $\mathcal{E}$, and $\mathscr{L}^2(G, \mathbb{C})$ is taken over by $\mathscr{L}^2(G, B)$. Letting γ denote a strongly continuous unitary action of G on $\mathcal{E}$ that respects the B-action on $\mathcal{E}$ (i.e., γ turns $\mathcal{E}$ into a Hilbert (G, B)-module), Meyer then calls an element $\xi \in \mathcal{E}$ **square-integrable** iff

$$\begin{Bmatrix} G & \to & B \\ x & \mapsto & \langle\gamma_x(\xi)|\eta\rangle_\mathcal{E} \end{Bmatrix} \in \mathscr{L}^2(G, B)$$

for all $\eta \in \mathcal{E}$. Clearly, the notion of 'square-integrability' parallels that of 'U-boundedness'.

Similarly, a unitary Hilbert-module representation $(\gamma, \mathcal{E})$ of G is said to be **square-integrable** iff the set of square-integrable elements is dense in $\mathcal{E}$. We can then generalize the theorem above to Hilbert modules to say that a unitary representation of G on a Hilbert module $\mathcal{E}$ is square-integrable iff the corresponding action of G on $\mathbb{K}(\mathcal{E})$ is proper. As a consequence, an action α on a C^*-algebra

B is integrable iff B, viewed as a Hilbert module over itself, is square-integrable (the action α clearly makes B a Hilbert (G, B)-module).

In [5], Meyer constructs a generalized fixed-point algebra for an ordinary C^*-dynamical system (G, B, β) as follows. He considers a Hilbert (G, B)-module $\mathcal{E}$. Then, to each element $\xi \in \mathcal{E}$, he associates two B-linear $\langle\!\langle \xi| : \mathcal{E} \to C_b(G, B)$ and $|\xi\rangle\!\rangle : C_c(G, B) \to \mathcal{E}$, called the **bra** and **ket** of ξ respetively, by

$$\langle\!\langle \xi|\eta = \begin{Bmatrix} G & \to & B \\ x & \mapsto & \langle \gamma_x(\xi)|\eta\rangle_{\mathcal{E}} \end{Bmatrix} \qquad \text{and} \qquad |\xi\rangle\!\rangle f = \int_G \gamma_x(\xi) \cdot f(x) \; \mathrm{d}x.$$

for all $\eta \in \mathcal{E}$ and $f \in C_c(G, B)$. By definition, ξ is square-integrable iff $\langle\!\langle \xi|\eta \in C_b(G, B) \cap L^2(G, B)$ for all $\eta \in \mathcal{E}$. Letting $\mathcal{E}_{\mathrm{si}}$ denote the set of all square-integrable elements of $\mathcal{E}$, Meyer then shows that $\xi \in \mathcal{E}_{\mathrm{si}}$ iff there is an extension of $|\xi\rangle\!\rangle$ to an adjointable Hilbert B-module operator $|\xi\rangle\!\rangle^{\mathrm{e}} : L^2(G, B) \to \mathcal{E}$. Now, suppose that there exists a dense linear subspace $\mathcal{R}$ of $\mathcal{E}$, called a **relatively continuous subspace**, satisfying

- $\mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$ and

- $\langle\!\langle \mathcal{R}|\mathcal{R}\rangle\!\rangle^{\mathrm{e}} := \langle\!\langle \mathcal{R}| \circ |\mathcal{R}\rangle\!\rangle^{\mathrm{e}} \subseteq C_r^*(G, B, \beta) \subseteq \mathbb{L}\big(L^2(G, B)\big)$.

Meyer turns $\mathcal{R}$ into a right $C_c(G, B)$-module (in the purely algebraic sense) that is (algebraically) isomorphic to $|\mathcal{R}\rangle\!\rangle^{\mathrm{e}}$ as a right $\rho[C_c(G, B)]$-module, where ρ denotes the right regular representation of $C_r^*(G, B, \beta)$. More precisely, the right action of $C_c(G, B)$ is given by

$$\xi * f = |\xi\rangle\!\rangle f.$$

As the ket operator is injective and $|\xi * f\rangle\!\rangle = |\xi\rangle\!\rangle \circ \rho_f$, the assertion is established. The completion of $\mathcal{R}$ with respect to the norm induced by a $C_c(G, B)$-valued inner product then yields a Hilbert $C_r^*(G, B, \beta)$-module $\mathcal{F}$. The generalized fixed-point algebra is then defined as the set of compact operators on $\mathcal{F}$, which is Morita-Rieffel equivalent to an ideal of $C_r^*(G, B, \beta)$, as can be seen using standard results in the theory of Morita-Rieffel equivalence. We thus end up with an imprimitivity $(\mathbb{K}(\mathcal{F}), J)$-bimodule $_{\mathbb{K}(\mathcal{F})}\mathcal{F}_J$.

Given a proper action α on A and a dense $*$-subalgebra A_0 guaranteed by the definition of properness, Rieffel proves that $A_0 \subseteq A_{\mathrm{si}}$ if we view A as a Hilbert (G, A)-module via the action α. It is clear from the properties listed above that $\langle\!\langle A_0|A_0\rangle\!\rangle^{\mathrm{e}} \subseteq L^1(G, A) \subseteq C_r^*(G, A, \alpha)$, which means that one can execute Meyer's procedure to construct a fixed-point algebra for the pair (A, A_0). We obtain an imprimitivty bimodule $_{\mathbb{K}(\mathcal{F})}\mathcal{F}_J$ that is actually dual to Rieffel's $_E\overline{A_0}_D$, and it turns out that Rieffel's fixed-point algebra D is anti-isomorphic to Meyer's fixed-point algebra $\mathbb{K}(\mathcal{F})$.

This is the precise connection between Rieffel's and Meyer's theories.

Meyer's beautiful and flexible framework has the potential to be generalized along several avenues. His PhD student Alcides Buss, for example, has recently adapted it to co-actions of locally compact quantum groups.

In fact, in [9], Rieffel himself had already hinted at the possibility of defining proper (co-)actions of quantum groups, namely Kac algebras. He also mentioned the possibility of defining proper actions of groupoids on C^*-algebras, and Jonathan Brown has provided just such a generalization in his 2009 PhD thesis [1]. Despite the recent spate of generalizations, as far as I know, it appears that nobody has considered a third avenue, which is to define **proper twisted actions** of groups on C^*-algebras.

My interest in developing this generalization originates from an observation that I made in [9]. There, Rieffel considered an action α of a group G on the C^*-algebra $\mathbb{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$. As is well-known, a unitary representation U of G on $\mathcal{H}$ yields an action $\mathrm{Ad}\, U$ of G on $\mathbb{K}(\mathcal{H})$. However, an action of G on $\mathbb{K}(\mathcal{H})$ is not necessarily equal to $\mathrm{Ad}\, U$ for any unitary

representation U of G on $\mathcal{H}$. Rieffel was well aware of this obstacle, so in order to circumvent it, he had to lift an action α of G on $\mathbb{K}(\mathcal{H})$ to an action $\tilde{\alpha}$ of G_α on $\mathbb{K}(\mathcal{H})$, where G_α is the central extension of G by $\mathbb{T}$. Then $\tilde{\alpha}$ comes from a genuine unitary representation of G_α on $\mathcal{H}$.

I felt that one could avoid this step if one uses the fact that any action α of G on $\mathcal{H}$ is equal to $\operatorname{Ad} \pi$ for some ω-representation π of G on $\mathcal{H}$, where $\omega : G \times G \to \mathbb{T}$ is a multiplier on G. It then follows that (G, A, α, ω) is a twisted C^*-dynamical system, and using the obvious action of $\mathbb{K}(\mathcal{H})$ on $\mathcal{H}$ itself, one had a twisted covariant representation of (G, A, α, ω) on $\mathcal{H}$.

Although central extensions are a standard tool in the theory of multiplier representations, I decided to investigate how one could directly handle a multiplier representation of a group without having to pass to an action of its central extension in order to get a unitary representation. Furthermore, if we consider more general twisted C^*-dynamical systems (G, B, β, ω), where β is no longer a genuine group action (unlike the case of α above) and ω takes values, not in $\mathbb{T}$, but in the multiplier algebra $M(B)$, then it appears that the notion of a central extension is rendered inapplicable.

Our aim here is to adapt Meyer's framework to twisted C^*-dynamical systems. Our ultimate goal, therefore, is to construct, for a given representation of a twisted C^*-dynamical system (G, B, β, ω) on a Hilbert B-module and under certain favorable conditions, a generalized fixed-point algebra that lies inside the reduced twisted crossed-product C^*-algebra corresponding to (G, B, β, ω).

It has been shown by Ruy Exel that the Packer-Raeburn stabilization trick allows us to view a *full* twisted crossed-product C^*-algebra as the cross-sectional C^*-algebra of a suitable Fell bundle. However, as our focus is on a *reduced* twisted crossed-product C^*-algebra, it is currently unclear how the theory of Fell bundles might help to simplify the content of this paper.

As much of this paper closely parallels [5], we must emphasize that we do not claim originality for the ideas contained herein, other than the simple act of introducing a twisted C^*-dynamical system into Meyer's framework. For the sake of convenience, we will reproduce many of Meyer's proofs and also provide more details where necessary. In the presence of twisting, certain measure-theoretical issues will appear that will inevitably force us to reformulate some of his results, so this paper should not be viewed as an exact replica of [5]. For example, the notion of 'G-equivariance' must be replaced by that of 'twisted G-equivariance'. Such issues clearly do not show up in [5], and some of them, I believe, might generate independent interest.

2. Vector-Valued Integration

Let (X, Σ, μ) be a σ-finite measure space and B a Banach space.

Let Σ_f denote the subset of Σ consisting of all Σ-measurable subsets of X with finite μ-measure.

- A function $\sigma : X \to B$ is called simple iff there exists a finite subset F of $\Sigma \times B$ such that
$$\sigma = \sum_{(E,b)\in F} \chi_E \cdot b.$$
We call F a datum for σ. Note that a datum for σ is not necessarily unique.

- A simple function $\sigma : X \to B$ is called integrable iff there exists a datum F for σ such that $F \subseteq \Sigma_f \times B$, in which case we define the integral of σ by
$$\int_G \sigma \, \mathrm{d}\mu := \sum_{(E,b)\in F} \mu(E) \cdot b \in B.$$
The integral of σ does not depend on how we choose a datum F for σ.

- A function $f : X \to B$ is called Bochner-measurable iff it is the almost-everywhere pointwise limit of a sequence of integrable simple functions from X to B.

- A function $f : X \to B$ is called Borel-measurable iff the pre-image of every open subset of B is a Σ-measurable subset of X.

- In the appendix, we will show that $f : X \to B$ is Bochner-measurable iff it is Borel-measurable. Therefore, if f is measurable in either sense, then we simply say that it is 'measurable'.

- If $f : X \to B$ is measurable, then $\begin{Bmatrix} G & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \|f(x)\|_B \end{Bmatrix}$ is also measurable.

- A measurable function $f : X \to B$ is called Bochner-integrable iff there exists a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of integrable simple functions from X to B, called an approximating sequence for f, such that
$$\lim_{n \to \infty} \int_G \|f - \sigma_n\|_B \; \mathrm{d}\mu = 0,$$
in which case we define the integral of f by
$$\int_G f \; \mathrm{d}\mu := \lim_{n \to \infty} \int_G \sigma_n \; \mathrm{d}\mu \in B.$$
The limit on the right-hand side exists and does not depend on how we choose an approximating sequence for f.

- A measurable function $f : X \to B$ is called null iff it assumes the value 0_B almost everywhere. We denote the set of all null functions from X to B by $\mathscr{N}_{X \to B}$.

Let G be a locally compact Hausdorff group. Denote its Borel σ-algebra by Σ_G, and let μ denote a Haar measure on G.

The subset of Σ_G consisting of those measurable subsets of G with finite μ-measure is denoted by $\Sigma_{G,\mu}^f$. This subset is independent of μ because all Haar measures on G are unique up to a positive scalar multiple. As such, we will denote it by Σ_G^f instead.

3. Twisted C^*-Dynamical Systems

Definition 1. A twisted C^*-dynamical system is a quadruple (G, B, β, ω), where:

- G is a second-countable, locally compact Hausdorff group.

- B is a C^*-algebra.

- $\beta : G \to \mathrm{Aut}(B)$ is a strongly measurable mapping, i.e.,
$$\begin{Bmatrix} G & \to & B \\ s & \mapsto & \beta_s(b) \end{Bmatrix}$$
is measurable for each $b \in B$.

- $\omega : G \times G \to U(M(B))$ is a strictly measurable mapping, i.e.,
$$\begin{Bmatrix} G \times G & \to & B \\ (s,t) & \mapsto & \omega(s,t) \, b \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} G \times G & \to & B \\ (s,t) & \mapsto & b \, \omega(s,t) \end{Bmatrix}$$
are measurable for each $b \in B$.

- $\beta_e = \mathrm{Id}_B$, and $\omega(e,s) = 1_{M(B)} = \omega(s,e)$ for all $s \in G$.

- $\overline{\beta_s} \circ \overline{\beta_t} = \mathrm{Ad}\,\omega(s,t) \circ \overline{\beta_{st}}$ for all $s,t \in G$.

- $\overline{\beta_r(\omega(s,t))} \, \omega(r,st) = \omega(r,s) \, \omega(rs,t)$ for all $r,s,t \in G$.

The symbol $\overline{\beta_s}$ denotes the unique extension of β_s to an element of $\mathrm{Aut}(M(B))$.

Note: When ω takes values in $\mathbb{T} \cdot 1_{M(B)}$, then the conditions

$$\overline{\beta_s} \circ \overline{\beta_t} = \mathrm{Ad}\ \omega(s,t) \circ \overline{\beta_{st}} \qquad \text{and} \qquad \overline{\beta_r(\omega(s,t))}\ \omega(r,st) = \omega(r,s)\ \omega(rs,t)$$

simply become

$$\overline{\beta_s} \circ \overline{\beta_t} = \overline{\beta_{st}} \qquad \text{and} \qquad \omega(s,t)\ \omega(r,st) = \omega(r,s)\ \omega(rs,t)$$

respectively. ♠

In this paper, (G, B, β, ω) is always a twisted C^*-dynamical system.

4. Twisted Hilbert C^*-Modules

Definition 2. A Hilbert (G, B, β, ω)-module is a $\mathbb{C}$-vector space $\mathcal{E}$ equipped with the following:

- A right B-action $. : \mathcal{E} \times B \to \mathcal{E}$.

- A positive-definite B-valued inner product $\langle \cdot | \cdot \rangle_{\mathcal{E}}$ such that $\mathcal{E}$ is complete with respect to $\|\cdot\|_{\mathcal{E}}$, the norm on $\mathcal{E}$ induced by $\langle \cdot | \cdot \rangle_{\mathcal{E}}$.

  Hence, $\mathcal{E}$ is a Hilbert B-module, and the automatic non-degeneracy of $.$ guarantees its extension to a right action of $M(B)$ on $\mathcal{E}$. By an abuse of notation, we denote the extended action by the same symbol.

- A strongly measurable mapping $\gamma : G \to \mathrm{Isom}(\mathcal{E})$, called a *twisted G-action*, that satisfies:

  (1) $\gamma_s(\zeta . b) = \gamma_s(\zeta) . \beta_s(b)$ for all $s \in G$, $\zeta \in \mathcal{E}$ and $b \in B$.

  (2) $\langle \gamma_s(\zeta) | \gamma_s(\eta) \rangle_{\mathcal{E}} = \beta_s(\langle \zeta | \eta \rangle_{\mathcal{E}})$ for all $s \in G$ and $\zeta, \eta \in \mathcal{E}$.

  (3) $\gamma_r(\gamma_s(\zeta)) = \gamma_{rs}(\zeta) . \omega(r,s)^*$ for all $r, s \in G$ and $\zeta \in \mathcal{E}$.

  By 'strongly measurable', we mean that

$$\left\{ \begin{array}{ccc} G & \to & \mathcal{E} \\ s & \mapsto & \gamma_s(\zeta) \end{array} \right\}$$

  is measurable for each $\zeta \in \mathcal{E}$.

We call the triple $(., \langle \cdot | \cdot \rangle_{\mathcal{E}}, \gamma)$ the structure of $\mathcal{E}$.

As (G, B, β, ω) is the only C^*-dynamical system that we will see in this paper, we will simply call any Hilbert (G, B, β, ω)-module a twisted Hilbert module. ♠

In this paper, $\mathcal{E}$ and $\mathcal{F}$ always denote twisted Hilbert modules. Without any risk of confusion, we will use the same symbol $.$ to denote their right B-actions. If, however, there is a need to distinguish their twisted G-actions, then we will use $\gamma^{\mathcal{E}}$ and $\gamma^{\mathcal{F}}$ respectively.

Definition 3. A mapping $T : \mathcal{E} \to \mathcal{F}$ (not assumed to be $\mathbb{C}$-linear) is called adjointable iff there exists a mapping $S : \mathcal{F} \to \mathcal{E}$, called an adjoint of T, such that

$$\langle T(\zeta) | \eta \rangle_{\mathcal{F}} = \langle \zeta | S(\eta) \rangle_{\mathcal{E}}$$

for all $\zeta \in \mathcal{E}$ and $\eta \in \mathcal{F}$. We denote the set of all adjointable mappings from $\mathcal{E}$ to $\mathcal{F}$ by $\mathbb{L}(\mathcal{E}, \mathcal{F})$, and those from $\mathcal{E}$ to itself by $\mathbb{L}(\mathcal{E})$. ♠

The following proposition shows that an adjointable mapping is automatically a Hilbert-module operator, i.e., a bounded B-module homomorphism. This is a classical result in the theory of Hilbert C^*-modules. Note, however, that not every Hilbert-module operator is adjointable [7].

Lemma 1. *An adjointable mapping $T : \mathcal{E} \to \mathcal{F}$ is a Hilbert-module operator.*

Proof. Fix $\zeta, \eta \in \mathcal{E}$ and $\lambda \in \mathbb{C}$. Then for all $\theta \in \mathcal{F}$, we have

$$\begin{aligned}
\langle T(\zeta + \lambda\eta)|\theta\rangle_{\mathcal{F}} &= \langle \zeta + \lambda\eta|S(\theta)\rangle_{\mathcal{E}} \\
&= \langle \zeta|S(\theta)\rangle_{\mathcal{E}} + \langle \lambda\eta|S(\theta)\rangle_{\mathcal{E}} \\
&= \langle \zeta|S(\theta)\rangle_{\mathcal{E}} + \lambda\langle \eta|S(\theta)\rangle_{\mathcal{E}} \\
&= \langle T(\zeta)|\theta\rangle_{\mathcal{F}} + \lambda\langle T(\eta)|\theta\rangle_{\mathcal{F}} \\
&= \langle T(\zeta)|\theta\rangle_{\mathcal{F}} + \langle \lambda \cdot T(\eta)|\theta\rangle_{\mathcal{F}} \\
&= \langle T(\zeta) + \lambda \cdot T(\eta)|\theta\rangle_{\mathcal{F}}.
\end{aligned}$$

Therefore, $T(\zeta + \lambda\eta) = T(\zeta) + \lambda \cdot T(\eta)$, so T is $\mathbb{C}$-linear.

Fix $\zeta, \eta \in \mathcal{E}$ as before and $b \in B$. Then for all $\theta \in \mathcal{F}$, we have

$$\begin{aligned}
\langle T(\zeta + \eta \cdot b)|\theta\rangle_{\mathcal{F}} &= \langle \zeta + \eta \cdot b|S(\theta)\rangle_{\mathcal{E}} \\
&= \langle \zeta|S(\theta)\rangle_{\mathcal{E}} + \langle \eta \cdot b|S(\theta)\rangle_{\mathcal{E}} \\
&= \langle \zeta|S(\theta)\rangle_{\mathcal{E}} + b^*\langle \eta|S(\theta)\rangle_{\mathcal{E}} \\
&= \langle T(\zeta)|\theta\rangle_{\mathcal{F}} + b^*\langle T(\eta)|\theta\rangle_{\mathcal{F}} \\
&= \langle T(\zeta)|\theta\rangle_{\mathcal{F}} + \langle T(\eta) \cdot b|\theta\rangle_{\mathcal{F}} \\
&= \langle T(\zeta) + T(\eta) \cdot b|\theta\rangle_{\mathcal{F}}.
\end{aligned}$$

Therefore, $T(\zeta + \eta \cdot b) = T(\zeta) + T(\eta) \cdot b$, so T is B-linear.

Finally, let $(\zeta_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{E}$ such that $(\zeta_n, T(\zeta_n))_{n \in \mathbb{N}}$ converges in $\mathcal{E} \times \mathcal{F}$ to some (ζ, η). By the Cauchy-Schwarz Inequality,

$$\begin{aligned}
\langle \eta|\theta\rangle_{\mathcal{F}} &= \lim_{n \to \infty} \langle T(\zeta_n)|\theta\rangle_{\mathcal{F}} \\
&= \lim_{n \to \infty} \langle \zeta_n|S(\theta)\rangle_{\mathcal{E}} \\
&= \langle \zeta|S(\theta)\rangle_{\mathcal{E}} \\
&= \langle T(\zeta)|\theta\rangle_{\mathcal{F}}
\end{aligned}$$

for all $\theta \in \mathcal{F}$. Therefore, $(\zeta, \eta) \in \mathrm{Graph}(T)$, and so T is bounded by the Closed Graph Theorem.　$\square$

5. $L^2(G, B)$ as a Twisted Hilbert C^*-Module

For each $p \in [1, \infty)$, let $\mathcal{L}^p(G, B)$ denote the $\mathbb{C}$-vector space of all measurable functions $\phi : G \to B$ such that

$$\int_G \|\phi(x)\|_B^p \, \mathrm{d}x < \infty.$$

Then let $L^p(G, B)$ denote the vector-space quotient of $\mathcal{L}^p(G, B)$ by the $\mathbb{C}$-linear subspace $\mathcal{N}_{G \to B}$. For each $\phi \in \mathcal{L}^p(G, B)$, we will denote its image in $L^p(G, B)$ by $[\![\phi]\!]$.

We now show that $L^2(G, B)$ can be equipped with a Hilbert (G, B, β, ω)-module structure. This is an important step in our generalization of Meyer's framework to twisted group actions.

The right B-action

The right action $\bullet$ of B on $L^2(G, B)$ is defined by

$$\forall \phi \in \mathcal{L}^2(G, B), \ \forall b \in B : \quad [\![\phi]\!] \bullet b := [\![x \mapsto \phi(x)b]\!].$$

It is easily checked that $\bullet$ is well-defined.

The positive-definite B-valued inner product

The positive-definite B-valued inner product $\langle\cdot|\cdot\rangle_2$ on $L^2(G,B)$ is defined by

$$\forall\phi_1,\phi_2 \in \mathscr{L}^2(G,B): \quad \langle[\![\phi_1]\!]|[\![\phi_2]\!]\rangle_2 := \int_G \phi_1(x)^*\phi_2(x) \ \mathrm{d}x.$$

We will verify the following statements in the appendix:

- The function $\begin{Bmatrix} G & \to & B \\ x & \mapsto & \phi_1(x)^*\phi_2(x) \end{Bmatrix}$ is Bochner-integrable for every $\phi_1,\phi_2 \in \mathscr{L}^2(G,B)$. This ensures that the integral above makes sense. Once we know this, it is then easily checked that $\langle\cdot|\cdot\rangle_2$ is well-defined.

- $L^2(G,B)$ is complete and separable with respect to $\|\cdot\|_2$.

- $\overline{L^2(G,B) \bullet B}^{\|\cdot\|_2} = L^2(G,B)$.

The twisted G-action

The twisted G-action $\Gamma: G \to \mathrm{Isom}\big(L^2(G,B)\big)$ is defined by

$$\forall s \in G, \ \forall\phi \in \mathscr{L}^2(G,B): \quad \Gamma_s([\![\phi]\!]) := \Big[\!\!\Big[x \mapsto \omega\big(s,s^{-1}x\big)^*\beta_s\big(\phi\big(s^{-1}x\big)\big)\Big]\!\!\Big].$$

We will verify the following statements in the appendix:

- $\Gamma_s([\![\phi]\!]) \in L^2(G,B)$ for each $\phi \in \mathscr{L}^2(G,B)$ and each $s \in G$.

- $\Gamma_s: L^2(G,B) \to L^2(G,B)$ is an adjointable isometric Hilbert-module operator for each $s \in G$.

- The function $\begin{Bmatrix} G & \to & L^2(G,B) \\ s & \mapsto & \Gamma_s([\![\phi]\!]) \end{Bmatrix}$ is measurable for each $\phi \in \mathscr{L}^2(G,B)$.

Let us now verify that Γ is indeed a twisted G-action:

(1) Let $s \in G$, $\phi \in \mathscr{L}^2(G,B)$ and $b \in B$. Then

$$\begin{aligned}
\Gamma_s([\![\phi]\!] \bullet b) &= \Gamma_s([\![x \mapsto \phi(x)b]\!]) \\
&= \Big[\!\!\Big[x \mapsto \omega\big(s,s^{-1}x\big)^*\beta_s\big(\phi\big(s^{-1}x\big)b\big)\Big]\!\!\Big] \\
&= \Big[\!\!\Big[x \mapsto \omega\big(s,s^{-1}x\big)^*\beta_s\big(\phi\big(s^{-1}x\big)\big)\beta_s(b)\Big]\!\!\Big] \\
&= \Big[\!\!\Big[x \mapsto \omega\big(s,s^{-1}x\big)^*\beta_s\big(\phi\big(s^{-1}x\big)\big)\Big]\!\!\Big] \bullet \beta_s(b) \\
&= \Gamma_s([\![\phi]\!]) \bullet \beta_s(b).
\end{aligned}$$

(2) Let $s \in G$ and $\phi,\psi \in \mathscr{L}^2(G,B)$. Then

$$\begin{aligned}
\langle\Gamma_s([\![\phi]\!])|\Gamma_s([\![\psi]\!])\rangle_2 &= \int_G \Big[\omega\big(s,s^{-1}x\big)^*\beta_s\big(\phi\big(s^{-1}x\big)\big)\Big]^*\omega\big(s,s^{-1}x\big)^*\beta_s\big(\psi\big(s^{-1}x\big)\big) \ \mathrm{d}x \\
&= \int_G \beta_s\big(\phi\big(s^{-1}x\big)\big)^*\omega\big(s,s^{-1}x\big)\omega\big(s,s^{-1}x\big)^*\beta_s\big(\psi\big(s^{-1}x\big)\big) \ \mathrm{d}x \\
&= \int_G \beta_s\big(\phi\big(s^{-1}x\big)\big)^*\beta_s\big(\psi\big(s^{-1}x\big)\big) \ \mathrm{d}x \\
&= \int_G \beta_s\big(\phi\big(s^{-1}x\big)^*\big)\beta_s\big(\psi\big(s^{-1}x\big)\big) \ \mathrm{d}x \\
&= \beta_s\left(\int_G \phi\big(s^{-1}x\big)^*\psi\big(s^{-1}x\big) \ \mathrm{d}x\right)
\end{aligned}$$

$$= \beta_s \left(\int_G \phi(x)^* \psi(x) \; \mathrm{d}x \right)$$

$$= \beta_s (\langle [\![\phi]\!] | [\![\psi]\!] \rangle_2).$$

(3) Let $r, s \in G$ and $\phi \in \mathscr{L}^2(G, B)$. Then

$$\Gamma_r(\Gamma_s([\![\phi]\!])) = \Gamma_r \left(\left[\!\left[x \mapsto \omega(s, s^{-1}x)^* \beta_s(\phi(s^{-1}x)) \right]\!\right] \right)$$

$$= \left[\!\left[x \mapsto \omega(r, r^{-1}x)^* \beta_r \left(\omega(s, s^{-1}r^{-1}x)^* \beta_s(\phi(s^{-1}r^{-1}x)) \right) \right]\!\right]$$

$$= \left[\!\left[x \mapsto \omega(r, r^{-1}x)^* \beta_r \left(\omega(s, s^{-1}r^{-1}x)^* \right) \beta_r(\beta_s(\phi(s^{-1}r^{-1}x))) \right]\!\right]$$

$$= \left[\!\left[x \mapsto \omega(r, r^{-1}x)^* \beta_r \left(\omega(s, s^{-1}r^{-1}x) \right)^* \beta_r(\beta_s(\phi(s^{-1}r^{-1}x))) \right]\!\right]$$

$$= \left[\!\left[x \mapsto \left[\beta_r\left(\omega(s, s^{-1}r^{-1}x) \right) \omega(r, r^{-1}x) \right]^* \omega(r, s) \beta_{rs}(\phi(s^{-1}r^{-1}x)) \omega(r, s)^* \right]\!\right]$$

$$= \left[\!\left[x \mapsto \left[\omega(r, s) \omega(rs, s^{-1}r^{-1}x) \right]^* \omega(r, s) \beta_{rs}(\phi(s^{-1}r^{-1}x)) \omega(r, s)^* \right]\!\right]$$

$$= \left[\!\left[x \mapsto \omega(rs, s^{-1}r^{-1}x)^* \omega(r, s)^* \omega(r, s) \beta_{rs}(\phi(s^{-1}r^{-1}x)) \omega(r, s)^* \right]\!\right]$$

$$= \left[\!\left[x \mapsto \omega(rs, s^{-1}r^{-1}x)^* \beta_{rs}(\phi(s^{-1}r^{-1}x)) \omega(r, s)^* \right]\!\right]$$

$$= \left[\!\left[x \mapsto \omega(rs, s^{-1}r^{-1}x)^* \beta_{rs}(\phi(s^{-1}r^{-1}x)) \right]\!\right] \bullet \omega(r, s)^*$$

$$= \Gamma_{rs}([\![\phi]\!]) \bullet \omega(r, s)^*.$$

6. Meyer's Bra-Ket Operators

Let $\mathscr{L}^\infty(G, B)$ denote the $\mathbb{C}$-vector space of all essentially bounded measurable functions from G to B. Next, let $\mathscr{L}^{\infty,c}(G, B)$ denote the $\mathbb{C}$-vector subspace of $\mathscr{L}^\infty(G, B)$ consisting of those essentially bounded measurable functions from G to B with support contained inside a compact subset of G.

We will denote by $L^\infty(G, B)$ and $L^{\infty,c}(G, B)$ respectively their vector-space quotients by $\mathscr{N}_{G \to B}$.

Let $\xi \in \mathcal{E}$. Define mappings $\langle\!\langle \xi | : \mathcal{E} \to L^\infty(G, B)$ and $|\xi\rangle\!\rangle : L^{\infty,c}(G, B) \to \mathcal{E}$ by

$$\forall \zeta \in \mathcal{E}, \; \forall x \in G : \quad \langle\!\langle \xi | \zeta := \left[\!\left[\left\{ \begin{matrix} G & \to & B \\ x & \mapsto & \langle \gamma_x(\xi) | \zeta \rangle_{\mathcal{E}} \end{matrix} \right\} \right]\!\right],$$

$$\forall f \in \mathscr{L}^{\infty,c}(G, B) : \quad |\xi\rangle\!\rangle [\![f]\!] := \int_G \gamma_x(\xi) \bullet f(x) \; \mathrm{d}x.$$

We call $\langle\!\langle \xi |$ the bra of ξ and $|\xi\rangle\!\rangle$ the ket of ξ. These mappings are due to Meyer, and they will serve a pivotal role in our theory as well.

An important result that we will need later on is the following, which has an easy proof in the framework of [5] but which in our setting, due to mappings not being continuous, requires more care.

Proposition 1. *Meyer's ket is injective, i.e., if $\xi \in \mathcal{E}$ and $|\xi\rangle\!\rangle : L^{\infty,c}(G, B) \to \mathcal{E}$ is the zero-mapping, then $\xi = 0_{\mathcal{E}}$.*

Proof. Suppose that $\xi \in \mathcal{E}$ satisfies the property that $|\xi\rangle\!\rangle : L^{\infty,c}(G, B) \to \mathcal{E}$ is the zero-mapping. As the mapping $F := \left\{ \begin{matrix} G & \to & \mathcal{E} \\ x & \mapsto & \gamma_x(\xi) \end{matrix} \right\}$ is by definition measurable, there exists a null subset N of G such that $F|_{G \setminus N}$ is the pointwise limit of a sequence of simple measurable functions from $G \setminus N$ to $\mathcal{E}$. In particular, the range of $F|_{G \setminus N}$ lies in a separable closed linear subspace $\mathcal{E}'$ of $\mathcal{E}$.

Let B' denote the separable C^*-subalgebra of B generated by $\langle \mathcal{E}'|\mathcal{E}'\rangle_{\mathcal{E}} \subseteq B$. Let $\mathcal{E}''$ denote the closed linear span of $\mathcal{E}' \cup (\mathcal{E}' \cdot B')$. We claim that $\mathcal{E}''$ is a separable Hilbert B'-module containing $\mathcal{E}'$. Clearly, $\mathcal{E}''$ is a separable and closed linear subspace of $\mathcal{E}$ that contains $\mathcal{E}'$. As

$$(\mathcal{E}' \cdot B') \cdot B' = \mathcal{E}' \cdot (B' \cdot B') = \mathcal{E}' \cdot B',$$

a limiting argument shows that $\mathcal{E}''$ is closed under the right action of B'. Also, as

$$\langle \mathcal{E}' \cdot B'|\mathcal{E}' \cdot B'\rangle_{\mathcal{E}} = (B')^* \langle \mathcal{E}'|\mathcal{E}'\rangle_{\mathcal{E}} B' = B'B'B' = B',$$

a limiting argument shows that $\langle \cdot|\cdot\rangle_{\mathcal{E}}$, when restricted to $\mathcal{E}'' \times \mathcal{E}''$, becomes a B'-valued inner product.

Now, the Hahn-Banach Theorem, together with the separability of $\mathcal{E}''$, gives us a countable family $\{\varphi_n\}_{n \in \mathbb{N}}$ of separating continuous linear functionals on $\mathcal{E}''$ such that

$$\|\xi\|_{\mathcal{E}} = \|\xi\|_{\mathcal{E}'} = \sup_{n \in \mathbb{N}}|\varphi_n(\xi)|$$

for all $\xi \in \mathcal{E}''$. Fixing an $m \in \mathbb{N}$, we have by continuity

$$\varphi_m(|\xi\rangle\!\rangle[\![f]\!]) = \varphi_m\left(\int_G \gamma_x(\xi) \cdot f(x)\ \mathrm{d}x\right) = \int_G \varphi_m(\gamma_x(\xi) \cdot f(x))\ \mathrm{d}x = 0$$

for all $f \in \mathscr{L}^{\infty,c}(G, B)$. In particular,

$$\int_E \varphi_m(\gamma_x(\xi) \cdot b)\ \mathrm{d}x = \int_G \varphi_m(\gamma_x(\xi) \cdot \chi_E(x)b)\ \mathrm{d}x = 0$$

for any $b \in B$ and any compact subset E of G.

Let $(e_j)_{j \in \mathbb{N}}$ be a countable approximate identity of B' (which exists because B' is separable). Then for any compact subset E of G, we get

$$\begin{aligned} 0 &= \lim_{j \to \infty} \int_E \varphi_m(\gamma_x(\xi) \cdot e_j)\ \mathrm{d}x \\ &= \int_E \left[\lim_{j \to \infty} \varphi_m(\gamma_x(\xi) \cdot e_j)\right]\ \mathrm{d}x \qquad \text{(By the LDCT.)} \\ &= \int_E \varphi_m(\gamma_x(\xi))\ \mathrm{d}x. \end{aligned}$$

The fact that $\mathcal{E}''$ is a Hilbert B'-module was used to ensure that $\lim_{j \to \infty} \gamma_x(\xi) \cdot e_j = \gamma_x(\xi)$ for all $x \in G \setminus N$, hence almost every $x \in G$.

By the σ-compactness of G, we see that $\varphi_m(\gamma_x(\xi)) = 0$ for almost every $x \in G$. The countable intersection of co-null sets is again co-null, so for almost every $x \in G$, we must have $\varphi_n(\gamma_x(\xi)) = 0$ for all $n \in \mathbb{N}$. Therefore, by the Hahn-Banach Theorem, $\gamma_x(\xi) = 0_{\mathcal{E}}$ for almost every $x \in G$. However, this implies $\xi = 0_{\mathcal{E}}$, which concludes the proof. □

Definition 4. An element $\xi \in \mathcal{E}$ is called square-integrable iff

$$\langle\!\langle \xi| : \mathcal{E} \to L^\infty(G, B) \cap L^2(G, B).$$

The set of square-integrable elements of $\mathcal{E}$ is denoted by $\mathcal{E}_{\mathrm{si}}$, and we say that $\mathcal{E}$ is square-integrable iff $\mathcal{E}_{\mathrm{si}}$ is dense in $\mathcal{E}$. ♠

In [5], Meyer adopts a slightly different definition of square-integrability. He defines the notion '$\langle\!\langle \xi|\zeta \in L^2(G, B)$' as: There exists a net $(\varphi_i)_{i \in I}$ in $C_c(G)^+$ converging to 1 on every compact subset of G such that the net $([\![\varphi_i]\!] \cdot \langle\!\langle \xi|\zeta)_{i \in I}$ converges in $L^2(G, B)$.

This naturally begs the question: Is there a difference between the two definitions? The answer is a rather subtle affirmative: For locally compact Hausdorff groups that are in a sense 'large' (i.e., those that are not σ-compact), the two definitions are inequivalent. For such large groups, an element

$\xi \in \mathcal{E}$ may be square-integrable in Meyer's sense even if $\langle\!\langle\xi|\zeta$ fails to be an element of $L^2(G, B)$ for some $\zeta \in \mathcal{E}$. Hence, Meyer's definition is weaker, but the ensuing generality allows his theory to work nicely for large locally compact Hausdorff groups.

In any case, it appears that we are forced to remain in the ream of second-countable (hence σ-compact) locally compact Hausdorff groups (for reasons that will be made apparent in the appendix), where both definitions coincide.

Lemma 2. *Let $\xi \in \mathcal{E}_{\mathrm{si}}$. Then $\left\langle [\![f]\!] \,\middle|\, \langle\!\langle\xi|\zeta\right\rangle_2 = \left\langle |\xi\rangle\!\rangle[\![f]\!] \,\middle|\, \zeta\right\rangle_{\mathcal{E}}$ for all $f \in \mathscr{L}^{\infty,c}(G, B)$ and $\zeta \in \mathcal{E}$.*

Proof. This is a matter of straightforward computation. $\qquad\square$

Proposition 2. *Let $\xi \in \mathcal{E}_{\mathrm{si}}$. Then $\langle\!\langle\xi| : \mathcal{E} \to L^2(G, B)$ is a Hilbert-module operator.*

Proof. The $\mathbb{C}$-linearity of ξ is clear from the definition. Let $\zeta, \eta \in \mathcal{E}$ and $b \in B$. Then

$$\begin{aligned}
\langle\!\langle\xi|(\zeta + \eta \bullet b) &= [\![x \mapsto \langle\gamma_x(\xi)|\zeta + \eta \bullet b\rangle_{\mathcal{E}}]\!] \\
&= [\![x \mapsto \langle\gamma_x(\xi)|\zeta\rangle_{\mathcal{E}} + \langle\gamma_x(\xi)|\eta \bullet b\rangle_{\mathcal{E}}]\!] \\
&= [\![x \mapsto \langle\gamma_x(\xi)|\zeta\rangle_{\mathcal{E}} + \langle\gamma_x(\xi)|\eta\rangle_{\mathcal{E}}b]\!] \\
&= [\![x \mapsto \langle\gamma_x(\xi)|\zeta\rangle_{\mathcal{E}}]\!] + [\![x \mapsto \langle\gamma_x(\xi)|\eta\rangle_{\mathcal{E}}b]\!] \\
&= [\![x \mapsto \langle\gamma_x(\xi)|\zeta\rangle_{\mathcal{E}}]\!] + [\![x \mapsto \langle\gamma_x(\xi)|\eta\rangle_{\mathcal{E}}]\!] \bullet b \\
&= \langle\!\langle\xi|\zeta + (\langle\!\langle\xi|\eta) \bullet b,
\end{aligned}$$

which proves that $\langle\!\langle\xi|$ is B-linear.

Let $(\zeta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ such that $(\zeta_n, \langle\!\langle\xi|\zeta_n)_{n\in\mathbb{N}}$ converges in $\mathcal{E} \times L^2(G, B)$ to some $(\zeta, [\![\phi]\!])$. Then for all $f \in \mathscr{L}^{\infty,c}(G, B)$, we have

$$\begin{aligned}
\langle[\![f]\!]|[\![\phi]\!]\rangle_2 &= \lim_{n\to\infty} \left\langle [\![f]\!] \,\middle|\, \langle\!\langle\xi|\zeta_n\right\rangle_2 \\
&= \lim_{n\to\infty} \left\langle |\xi\rangle\!\rangle[\![f]\!] \,\middle|\, \zeta_n\right\rangle_{\mathcal{E}} \qquad \text{(By Lemma 2.)} \\
&= \left\langle |\xi\rangle\!\rangle[\![f]\!] \,\middle|\, \zeta\right\rangle_{\mathcal{E}} \\
&= \left\langle [\![f]\!] \,\middle|\, \langle\!\langle\xi|\zeta\right\rangle_2. \qquad \text{(By Lemma 2 again.)}
\end{aligned}$$

As $L^{\infty,c}(G, B)$ is dense in $L^2(G, B)$, it follows that $(\zeta, [\![\phi]\!]) \in \mathrm{Graph}(\langle\!\langle\xi|)$. Therefore, $\langle\!\langle\xi|$ is bounded by the Closed Graph Theorem. $\qquad\square$

Proposition 3. *Let $\xi \in \mathcal{E}_{\mathrm{si}}$. Then:*

(1) $|\xi\rangle\!\rangle : L^{\infty,c}(G, B) \to \mathcal{E}$ can be uniquely extended to a Hilbert-module operator $T : L^2(G, B) \to \mathcal{E}$.

(2) $\langle\!\langle\xi| : \mathcal{E} \to L^2(G, B)$ is an adjointable Hilbert-module operator adjoint to T.

Proof. Let $\phi \in \mathscr{L}^2(G, B) \setminus \mathscr{L}^{\infty,c}(G, B)$. As $L^{\infty,c}(G, B)$ is dense in $L^2(G, B)$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{L}^{\infty,c}(G, B)$ such that $([\![f_n]\!])_{n\in\mathbb{N}}$ converges in $L^2(G, B)$ to $[\![\phi]\!]$. By the previous theorem, there exists a constant $C > 0$ such that

$$\|\langle\!\langle\xi|\zeta\|_2 \leq C\|\zeta\|_{\mathcal{E}}$$

for all $\zeta \in \mathcal{E}$. Fixing $m, n \in \mathbb{N}$, we then have

$$\begin{aligned}
&\||\xi\rangle\!\rangle([\![f_m - f_n]\!])\|_{\mathcal{E}}^2 \\
&= \left\|\left\langle |\xi\rangle\!\rangle([\![f_m - f_n]\!]) \,\middle|\, |\xi\rangle\!\rangle([\![f_m - f_n]\!])\right\rangle_{\mathcal{E}}\right\|_B
\end{aligned}$$

$$= \left\| \left\langle [\![f_m - f_n]\!] \ \middle| \ \langle\!\langle \xi | (|\xi\rangle\!\rangle ([\![f_m - f_n]\!])) \right\rangle_2 \right\|_B$$

$$\leq \| [\![f_m - f_n]\!] \|_2 \ \| \langle\!\langle \xi | (|\xi\rangle\!\rangle ([\![f_m - f_n]\!])) \|_2 \qquad \text{(By the Cauchy-Schwarz Inequality.)}$$

$$\leq \| [\![f_m - f_n]\!] \|_2 \cdot C \| |\xi\rangle\!\rangle ([\![f_m - f_n]\!]) \|_{\mathcal{E}}.$$

Hence,

$$\| |\xi\rangle\!\rangle ([\![f_m - f_n]\!]) \|_{\mathcal{E}} \leq C \| [\![f_m - f_n]\!] \|_2.$$

As $([\![f_n]\!])_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(G, B)$, it follows that $(|\xi\rangle\!\rangle [\![f_n]\!])_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}$. Then by the completeness of $\mathcal{E}$, we see that $(|\xi\rangle\!\rangle [\![f_n]\!])_{n \in \mathbb{N}}$ converges in $\mathcal{E}$ to a unique ζ_ϕ that depends only on $[\![\phi]\!]$. Defining $T([\![\phi]\!]) := \zeta_\phi$, we have

$$\left\langle T([\![\phi]\!]) \ \middle| \ \eta \right\rangle_{\mathcal{E}} = \lim_{n \to \infty} \left\langle |\xi\rangle\!\rangle [\![f_n]\!] \ \middle| \ \eta \right\rangle_{\mathcal{E}}$$

$$= \lim_{n \to \infty} \left\langle [\![f_n]\!] \ \middle| \ \langle\!\langle \xi | \eta \right\rangle_2$$

$$= \left\langle [\![\phi]\!] \ \middle| \ \langle\!\langle \xi | \eta \right\rangle_2$$

for all $\eta \in \mathcal{E}$. It follows that $|\xi\rangle\!\rangle : L^{\infty, c}(G, B) \to \mathcal{E}$ can be extended to a unique mapping $T : L^2(G, B) \to \mathcal{E}$ that is adjoint to $\langle\!\langle \xi |$. Therefore, T is adjointable, hence bounded by Lemma 1, which proves (1).

We immediately get (2) as the adjointability relation between two mappings is a symmetric one. $\square$

In summary, what Proposition 3 says is this: If $\xi \in \mathcal{E}_{\mathrm{si}}$, then $|\xi\rangle\!\rangle : L^{\infty, c}(G, B) \to \mathcal{E}$ can be extended to a unique adjointable Hilbert-module operator $T : L^2(G, B) \to \mathcal{E}$. We will denote this operator T by $|\xi\rangle\!\rangle^{\mathrm{e}}$.

The converse of this statement is also true, as the next theorem shows.

Theorem 2. *Let $\xi \in \mathcal{E}$. If $|\xi\rangle\!\rangle : L^{\infty, c}(G, B) \to \mathcal{E}$ can be extended to an adjointable Hilbert-module operator $T : L^2(G, B) \to \mathcal{E}$, then $\xi \in \mathcal{E}_{\mathrm{si}}$.*

Proof. Suppose we can extend $|\xi\rangle\!\rangle : L^{\infty, c}(G, B) \to \mathcal{E}$ to an adjointable Hilbert-module operator $T : L^2(G, B) \to \mathcal{E}$. Let $S : \mathcal{E} \to L^2(G, B)$ denote the adjoint of T.

Let K be a compact subset of G, and fix $\zeta \in \mathcal{E}$. Then for all $f \in \mathscr{L}^{\infty, c}(G, B)$, we have

$$\left\langle [\![f]\!] \ \middle| \ [\![\chi_K]\!] \cdot \langle\!\langle \xi | \zeta \right\rangle_2 = \left\langle [\![f]\!] \ \middle| \ [\![\chi_K]\!] \cdot [\![x \mapsto \langle \gamma_x(\xi) | \zeta \rangle_{\mathcal{E}}]\!] \right\rangle_2$$

$$= \int_G \chi_K(x) f(x)^* \langle \gamma_x(\xi) | \zeta \rangle_{\mathcal{E}} \ \mathrm{d}x$$

$$= \left\langle \int_G \gamma_x(\xi) \bullet \chi_K(x) f(x) \ \mathrm{d}x \ \middle| \ \zeta \right\rangle_{\mathcal{E}}$$

$$= \left\langle |\xi\rangle\!\rangle ([\![\chi_K \cdot f]\!]) \ \middle| \ \zeta \right\rangle_{\mathcal{E}}$$

$$= \left\langle T([\![\chi_K \cdot f]\!]) \ \middle| \ \zeta \right\rangle_{\mathcal{E}}$$

$$= \left\langle [\![\chi_K \cdot f]\!] \ \middle| \ S(\zeta) \right\rangle_2$$

$$= \left\langle [\![f]\!] \ \middle| \ [\![\chi_K]\!] \cdot S(\zeta) \right\rangle_2.$$

The denseness of $L^{\infty, c}(G, B)$ in $L^2(G, B)$ thus yields $[\![\chi_K]\!] \cdot \langle\!\langle \xi | \zeta = [\![\chi_K]\!] \cdot S(\zeta)$. Finally, we invoke the σ-compactness of (G, Σ_G, μ) to conclude that $\langle\!\langle \xi | \zeta$ and $S(\zeta)$ belong to the same measurability class, which proves that $\langle\!\langle \xi | \zeta \in L^2(G, B)$. $\square$

The last step of the proof might break down if (G, Σ_G, μ) is not σ-compact because a union of uncountably many null subsets of G might fail to be null. If, however, we are willing to accept Martin's Axiom for an infinite cardinal κ (denoted by $\mathrm{MA}(\kappa)$), where $\kappa < \mathfrak{c}$, then the proof will hold for all locally compact Hausdorff groups that are a union of at most κ-many compact subsets. Of course, this is interesting only if the Continuum Hypothesis were false, otherwise the only infinite cardinal $< \mathfrak{c}$ would be $\aleph_0$ itself.

Although $\mathcal{E}_{\mathrm{si}}$ may not be complete with respect to the norm $\|\cdot\|_\mathcal{E}$, it is complete with respect to the norm $\|\cdot\|_{\mathrm{si}}$ defined by

(1) $$\forall \xi \in \mathcal{E}_{\mathrm{si}}: \quad \|\xi\|_{\mathrm{si}} := \|\xi\|_\mathcal{E} + \||\xi\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}(L^2(G,B),\mathcal{E})}$$

(2) $$= \|\xi\|_\mathcal{E} + \sqrt{\||\langle\!\langle\xi|\xi\rangle\!\rangle\|_{\mathbb{L}(L^2(G,B))}}. \qquad (\text{As } |\xi\rangle\!\rangle^* = \langle\!\langle\xi|.)$$

Let us give a formal proof of this assertion.

Proposition 4. $\mathcal{E}_{\mathrm{si}}$, when equipped with the norm $\|\cdot\|_{\mathrm{si}}$, is a Banach space.

Proof. Let $(\xi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{E}_{\mathrm{si}}$ with respect to $\|\cdot\|_{\mathrm{si}}$. Then it is a Cauchy sequence in $\mathcal{E}$ with respect to $\|\cdot\|_\mathcal{E}$, and so by completeness, it possesses a $\|\cdot\|_{\mathrm{si}}$-limit $\xi \in \mathcal{E}$. Observe also that $(|\xi_n\rangle\!\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}(L^2(G,B),\mathcal{E})$ with respect to the operator norm. By completeness once again, this sequence has a limit $T \in \mathbb{L}(L^2(G,B),\mathcal{E})$ with respect to the operator norm. Then for all $f \in \mathscr{L}^{\infty,c}(G,B)$, we have

$$\begin{aligned}
T([\![f]\!]) &= \lim_{n\to\infty} |\xi_n\rangle\!\rangle[\![f]\!] \\
&= \lim_{n\to\infty} \int_G \gamma_x(\xi_n) \cdot f(x) \, \mathrm{d}x \\
&= \int_G \left[\lim_{n\to\infty} \gamma_x(\xi_n) \cdot f(x) \right] \mathrm{d}x \qquad (\text{By the LDCT.}) \\
&= \int_G \gamma_x(\xi) \cdot f(x) \, \mathrm{d}x \\
&= |\xi\rangle\!\rangle[\![f]\!],
\end{aligned}$$

where convergence is with respect to $\|\cdot\|_\mathcal{E}$. Hence, $|\xi\rangle\!\rangle$ can be extended to the adjointable Hilbert-module operator $T : L^2(G,B) \to \mathcal{E}$. By Theorem 2, we therefore obtain $\xi \in \mathcal{E}_{\mathrm{si}}$, and it follows readily that $(\xi_n)_{n\in\mathbb{N}}$ converges to ξ with respect to $\|\cdot\|_{\mathrm{si}}$. $\qquad\square$

7. Twisted G-Equivariance

Definition 5. A Hilbert-module operator $T : \mathcal{E} \to \mathcal{F}$ is called twisted G-equivariant iff

$$\gamma_s^\mathcal{F}(T(\zeta)) = T(\gamma_s^\mathcal{E}(\zeta))$$

for all $s \in G$ and $\zeta \in \mathcal{E}$. ♠

Let $\xi \in \mathcal{E}_{\mathrm{si}}$. In [5], the bra-ket operators were obviously G-equivariant, so in our setting, we would naturally expect $\langle\!\langle\xi| : \mathcal{E} \to L^2(G,B)$ and $|\xi\rangle\!\rangle^{\mathrm{e}} : L^2(G,B) \to \mathcal{E}$ to be *twisted G-equivariant*. However, this is no longer an obvious claim and requires some demonstration.

Let $s \in G$ and $\zeta \in \mathcal{E}$. Then

$$\begin{aligned}
\Gamma_s(\langle\!\langle\xi|\zeta) &= \Gamma_s([\![x \mapsto \langle\gamma_x(\xi)|\zeta\rangle_\mathcal{E}]\!]) \\
&= \left[\!\left[x \mapsto \omega(s, s^{-1}x)^* \, \beta_s(\langle\gamma_{s^{-1}x}(\xi)|\zeta\rangle_\mathcal{E}) \right]\!\right] \\
&= \left[\!\left[x \mapsto \omega(s, s^{-1}x)^* \, \langle\gamma_s(\gamma_{s^{-1}x}(\xi))|\gamma_s(\zeta)\rangle_\mathcal{E} \right]\!\right]
\end{aligned}$$

$$= \left[\!\!\left[x \mapsto \omega\big(s, s^{-1}x\big)^* \left\langle \gamma_x(\xi) \cdot \omega\big(s, s^{-1}x\big)^* \Big| \gamma_s(\zeta) \right\rangle_{\mathcal{E}} \right]\!\!\right]$$

$$= \left[\!\!\left[x \mapsto \omega\big(s, s^{-1}x\big)^* \, \omega\big(s, s^{-1}x\big) \, \langle \gamma_x(\xi) | \gamma_s(\zeta) \rangle_{\mathcal{E}} \right]\!\!\right]$$

$$= \left[\!\!\left[x \mapsto \langle \gamma_x(\xi) | \gamma_s(\zeta) \rangle_{\mathcal{E}} \right]\!\!\right]$$

$$= \langle\!\langle \xi | (\gamma_s(\zeta)).$$

Let $s \in G$ and $f \in \mathscr{L}^{\infty,c}(G, B)$. Then

$$\gamma_s(|\xi\rangle\!\rangle^{\mathrm{e}}[\![f]\!]) = \gamma_s(|\xi\rangle\!\rangle[\![f]\!])$$

$$= \gamma_s\left(\int_G \gamma_x(\xi) \cdot f(x) \, \mathrm{d}x \right)$$

$$= \int_G \gamma_s(\gamma_x(\xi) \cdot f(x)) \, \mathrm{d}x$$

$$= \int_G \gamma_s(\gamma_x(\xi)) \cdot \beta_s(f(x)) \, \mathrm{d}x$$

$$= \int_G \left[\gamma_{sx}(\xi) \cdot \omega(s, x)^* \right] \cdot \beta_s(f(x)) \, \mathrm{d}x$$

$$= \int_G \gamma_{sx}(\xi) \cdot \left[\omega(s, x)^* \, \beta_s(f(x)) \right] \, \mathrm{d}x$$

$$= \int_G \gamma_x(\xi) \cdot \left[\omega\big(s, s^{-1}x\big)^* \, \beta_s\big(f\big(s^{-1}x\big)\big) \right] \, \mathrm{d}x \qquad \text{(By a change of variables.)}$$

$$= |\xi\rangle\!\rangle\left(\left[\!\!\left[x \mapsto \omega\big(s, s^{-1}x\big)^* \, \beta_s\big(f\big(s^{-1}x\big)\big) \right]\!\!\right] \right)$$

$$= |\xi\rangle\!\rangle(\Gamma_s([\![f]\!]))$$

$$= |\xi\rangle\!\rangle^{\mathrm{e}}(\Gamma_s([\![f]\!])).$$

Now, use the density of $L^{\infty,c}(G, B)$ in $L^2(G, B)$ to show that $\gamma_s(|\xi\rangle\!\rangle^{\mathrm{e}}[\![f]\!]) = |\xi\rangle\!\rangle^{\mathrm{e}}(\Gamma_s([\![f]\!]))$ for all $f \in \mathscr{L}^2(G, B)$.

In fact, it suffices to only prove that $\langle\!\langle \xi |$ is twisted G-equivariant, thanks to the following result.

Theorem 3. *If an adjointable Hilbert-module operator $T : \mathcal{E} \to \mathcal{F}$ is twisted G-equivariant, then $S : \mathcal{F} \to \mathcal{E}$ is also a twisted G-equivariant Hilbert-module operator, where S is the adjoint of T.*

Proof. Let $s \in G$ and $b \in B$. Then for all $\xi \in \mathcal{E}$ and $\eta \in \mathcal{F}$, we have

$$\left\langle S\big(\gamma_s^{\mathcal{F}}(\eta)\big) \big| \xi \cdot b \right\rangle_{\mathcal{E}} = \left\langle \gamma_s^{\mathcal{F}}(\eta) \big| T(\xi \cdot b) \right\rangle_{\mathcal{F}}$$

$$= \left\langle \gamma_s^{\mathcal{F}}(\eta) \big| T(\xi) \cdot b \right\rangle_{\mathcal{F}} \qquad \text{(As } T \text{ is } B\text{-linear.)}$$

$$= \left\langle \gamma_s^{\mathcal{F}}(\eta) \big| \gamma_s^{\mathcal{F}}\big(\gamma_{s^{-1}}^{\mathcal{F}}(T(\xi))\big) \cdot \omega\big(s, s^{-1}\big) \, b \right\rangle_{\mathcal{F}}$$

$$= \left\langle \gamma_s^{\mathcal{F}}(\eta) \big| \gamma_s^{\mathcal{F}}\big(\gamma_{s^{-1}}^{\mathcal{F}}(T(\xi))\big) \right\rangle_{\mathcal{F}} \omega\big(s, s^{-1}\big) \, b$$

$$= \beta_s\big(\big\langle \eta \big| \gamma_{s^{-1}}^{\mathcal{F}}(T(\xi)) \big\rangle_{\mathcal{F}}\big) \, \omega\big(s, s^{-1}\big) \, b$$

$$= \beta_s\big(\big\langle \eta \big| T\big(\gamma_{s^{-1}}^{\mathcal{E}}(\xi)\big) \big\rangle_{\mathcal{F}}\big) \, \omega\big(s, s^{-1}\big) \, b \qquad \text{(As } T \text{ is } G\text{-equivariant.)}$$

$$= \beta_s\big(\big\langle S(\eta) \big| \gamma_{s^{-1}}^{\mathcal{E}}(\xi) \big\rangle_{\mathcal{E}}\big) \, \omega\big(s, s^{-1}\big) \, b$$

$$= \left\langle \gamma_s^{\mathcal{E}}(S(\eta)) \big| \gamma_s^{\mathcal{E}}\big(\gamma_{s^{-1}}^{\mathcal{E}}(\xi)\big) \right\rangle_{\mathcal{E}} \omega\big(s, s^{-1}\big) \, b$$

$$= \left\langle \gamma_s^{\mathcal{E}}(S(\eta)) \big| \xi \cdot \omega\big(s, s^{-1}\big)^* \right\rangle_{\mathcal{E}} \omega\big(s, s^{-1}\big) \, b$$

$$= \left\langle \gamma_s^{\mathcal{E}}(S(\eta)) \big| \xi \cdot \omega\big(s, s^{-1}\big)^* \, \omega\big(s, s^{-1}\big) \, b \right\rangle_{\mathcal{E}}$$

$$= \big\langle \gamma_s^{\mathcal{E}}(S(\eta)) \big| \xi \cdot b \big\rangle_{\mathcal{E}}.$$

As $\xi, \eta \in \mathcal{E}$ and $b \in B$ are arbitrary, and as $\cdot$ is non-degenerate, we obtain

$$S\big(\gamma_s^{\mathcal{F}}(\eta)\big) = \gamma_s^{\mathcal{E}}(S(\eta)).$$

Therefore, S is twisted G-equivariant. $\qquad\square$

Definition 6. The set of all twisted G-equivariant adjointable Hilbert-module operators from $\mathcal{E}$ to $\mathcal{F}$ is denoted by $\mathbb{L}^G(\mathcal{E}, \mathcal{F})$, and those from $\mathcal{E}$ to itself by $\mathbb{L}^G(\mathcal{E})$. $\qquad\spadesuit$

8. Reduced (Busby-Smith) Twisted Crossed Products

We can make $L^1(G, B)$ into a convolution Banach $*$-algebra via the following operations:

$$\forall K, L \in \mathscr{L}^1(G, B): \quad [\![K]\!] \star [\![L]\!] := \left[\!\!\left[x \mapsto \int_G K(y)\, \beta_y\big(L\big(y^{-1}x\big)\big)\, \omega\big(y, y^{-1}x\big)\, \mathrm{d}y \right]\!\!\right],$$

$$[\![K]\!]^* := \left[\!\!\left[x \mapsto \omega\big(x, x^{-1}\big)^*\, \beta_x\Big(K\big(x^{-1}\big)^*\Big)\, \Delta\big(x^{-1}\big) \right]\!\!\right],$$

where Δ denotes the modular function of G. Note that the involution is isometric with respect to the norm $\|\cdot\|_1$.

Next, define a mapping $\rho : L^1(G, B) \to \mathbb{L}^G\big(L^2(G, B)\big)$ by

$$\rho_{[\![K]\!]}([\![\phi]\!]) := \left[\!\!\left[x \mapsto \int_G \sqrt{\Delta(x^{-1}y)}\, \beta_x\big(K\big(x^{-1}y\big)\big)\, \omega\big(x, x^{-1}y\big)\, \phi(y)\, \mathrm{d}y \right]\!\!\right].$$

for all $K \in \mathscr{L}^1(G, B)$ and all $\phi \in \mathscr{L}^2(G, B)$. A tedious computation shows that ρ is actually a $*$-representation of $L^1(G, B)$ by twisted G-equivariant adjointable Hilbert-module operators on $L^2(G, B)$.

As ρ is a $*$-homomorphism from a Banach $*$-algebra to a C^*-algebra, it is automatically contractive:

$$\forall K \in \mathscr{L}^1(G, B): \quad \big\|\rho_{[\![K]\!]}\big\|_{\mathbb{L}^G(L^2(G,B))} \leq \big\|[\![K]\!]\big\|_1.$$

The completion of the image of $L^1(G, B)$ under ρ in $\mathbb{L}^G\big(L^2(G, B)\big)$ is then called the reduced twisted crossed product of the twisted C^*-dynamical system (G, B, β, ω), and it is denoted by $C_r^*(G, B, \beta, \omega)$.

As $L^{\infty,c}(G, B)$ is a dense linear subspace of $L^1(G, B)$ (with respect to the $\|\cdot\|_1$-norm), we also have that $C_r^*(G, B, \beta, \omega)$ is the completion of the image of $L^{\infty,c}(G, B)$ under ρ. Furthermore, it can be shown that $L^{\infty,c}(G, B)$ is a convolution $*$-algebra under $\star$. Hence, in our original definition of $C_r^*(G, B, \beta, \omega)$, we could have worked with $L^{\infty,c}(G, B)$ instead of $L^1(G, B)$.

If K happens to be a distribution on G that takes values in the multiplier algebra $M(B)$, then we can define ρ_K accordingly. For example, consider the Dirac-delta distributions $b\delta_1$ for $b \in B$ and δ_g for $g \in G$. We can define $\rho_b := \rho_{b\delta_1}$ and $\rho_g := \rho_{\delta_g}$ by

$$\forall \phi \in \mathscr{L}^2(G, B): \quad \rho_b([\![\phi]\!]) := [\![x \mapsto \beta_x(b)\, \phi(x)]\!],$$

$$\rho_g([\![\phi]\!]) := \left[\!\!\left[x \mapsto \sqrt{\Delta(g)}\, \omega(x, g)\, \phi(xg) \right]\!\!\right].$$

Note that ρ_b and ρ_g are both adjointable operators, and that ρ_g is furthermore an isometry.

The following ket identities are easily verified and will be required at a later stage.

$$(3) \qquad\qquad \forall T \in \mathbb{L}^G(\mathcal{E}, \mathcal{F}),\ \forall \xi \in \mathcal{E}: \quad |T(\xi)\rangle\!\rangle = T \circ |\xi\rangle\!\rangle;$$

$$(4) \qquad\qquad \forall \xi \in \mathcal{E},\ \forall b \in B: \quad |\xi \cdot b\rangle\!\rangle = |\xi\rangle\!\rangle \circ \rho_b|_{L^{\infty,c}(G,B)};$$

$$(5) \qquad\qquad \forall \xi \in \mathcal{E},\ \forall g \in G: \quad |\gamma_g(\xi)\rangle\!\rangle = |\xi\rangle\!\rangle \circ \rho_g^*|_{L^{\infty,c}(G,B)}.$$

The last identity clearly implies that $\mathcal{E}_{\mathrm{si}}$ is twisted G-invariant.

Lemma 3. *The following norm estimates hold for all* $T \in \mathbb{L}^G(\mathcal{E}, \mathcal{F})$, $\xi \in \mathcal{E}_{\mathrm{si}}$, $b \in B$ *and* $g \in G$:

$$\tag{6} \|T(\xi)\|_{\mathrm{si}} \leq \|\xi\|_{\mathrm{si}} \cdot \|T\|_{\mathbb{L}^G(\mathcal{E}, \mathcal{F})}.$$

$$\tag{7} \|\xi \cdot b\|_{\mathrm{si}} \leq \|\xi\|_{\mathrm{si}} \cdot \|b\|_B.$$

$$\tag{8} \|\gamma_g(\xi)\|_{\mathrm{si}} \leq \|\xi\|_{\mathrm{si}}.$$

Proof. These estimates follow from a straightforward argument using easy facts about operator norms. For the third estimate, we use the fact that ρ_g is an isometric adjointable operator to conclude that $\|\rho_g^*\|_{\mathbb{L}^G(L^2(G,B))} = 1$. $\qquad\square$

We now show how to make $\mathcal{E}$ into a right $L^{\infty,c}(G, B)$-module. For each $K \in \mathscr{L}^{\infty,c}(G, B)$, define

$$K^\vee := \left\{ \begin{matrix} G & \to & B \\ x & \mapsto & \sqrt{\Delta(x^{-1})}\, \omega(x, x^{-1})^* \, \beta_x(K(x^{-1})) \end{matrix} \right\} \in \mathscr{L}^{\infty,c}(G, B),$$

$$K^\wedge := \left\{ \begin{matrix} G & \to & B \\ x & \mapsto & \sqrt{\Delta(x^{-1})}\, \beta_x(K(x^{-1}))\, \omega(x, x^{-1}) \end{matrix} \right\} \in \mathscr{L}^{\infty,c}(G, B).$$

Then $(K^\vee)^\wedge = (K^\wedge)^\vee = K$, which means that $\vee$ and $\wedge$ are inverse operations on $\mathscr{L}^{\infty,c}(G, B)$.

Now, fix $K \in L^{\infty,c}(G, B)$. Viewing $L^{\infty,c}(G, B)$ as a subset of the twisted Hilbert module $L^2(G, B)$, and recalling the definition of the twisted G-action Γ on $L^2(G, B)$, we have

$$|[\![K]\!]\rangle\!\rangle[\![\phi]\!] = \int_G \Gamma_x([\![K]\!]) \cdot \phi(x) \; \mathrm{d}x$$

$$= \left[\!\!\left[x \mapsto \int_G \omega(y, y^{-1}x)\, \beta_y(K(y^{-1}x))\, \phi(y) \; \mathrm{d}y \right]\!\!\right]$$

$$= \rho_{[\![K^\vee]\!]}([\![\phi]\!])$$

$$\in L^{\infty,c}(G, B)$$

for all $\phi \in \mathscr{L}^{\infty,c}(G, B)$. It follows that $|[\![K]\!]\rangle\!\rangle : L^{\infty,c}(G, B) \to L^{\infty,c}(G, B)$ can be extended to the adjointable Hilbert-module operator $\rho_{[\![K^\vee]\!]} : L^2(G, B) \to L^2(G, B)$. Hence, $L^{\infty,c}(G, B) \subseteq L^2(G, B)_{\mathrm{si}}$ and $|[\![K^\wedge]\!]\rangle\!\rangle^{\mathrm{e}} = \rho_{[\![K]\!]}$.

Define $\xi * [\![K]\!] := |\xi\rangle\!\rangle[\![K^\wedge]\!] \in \mathcal{E}$ for each $\xi \in \mathcal{E}$. Then

$$|\xi * [\![K]\!]\rangle\!\rangle = |\xi\rangle\!\rangle \circ |[\![K^\wedge]\!]\rangle\!\rangle = |\xi\rangle\!\rangle \circ \rho_{[\![K]\!]}|_{L^{\infty,c}(G,B)}.$$

This yields

$$|(\xi * [\![K]\!]) * [\![L]\!]\rangle\!\rangle = |\xi * [\![K]\!]\rangle\!\rangle \circ \rho_{[\![L]\!]}|_{L^{\infty,c}(G,B)}$$

$$= |\xi\rangle\!\rangle \circ \rho_{[\![K]\!]}|_{L^{\infty,c}(G,B)} \circ \rho_{[\![L]\!]}|_{L^{\infty,c}(G,B)}$$

$$= |\xi\rangle\!\rangle \circ \rho_{[\![K\star L]\!]}|_{L^{\infty,c}(G,B)}$$

$$= |\xi * ([\![K]\!] \star [\![L]\!])\rangle\!\rangle.$$

By the injectivity of Meyer's ket (Proposition 1), we get $(\xi * [\![K]\!]) * [\![L]\!] = \xi * ([\![K]\!] \star [\![L]\!])$. Therefore, $*$ is a right $L^{\infty,c}(G, B)$-action on $\mathcal{E}$.

Lemma 4. *The following norm estimate holds for all* $\xi \in \mathcal{E}_{\mathrm{si}}$ *and* $K \in \mathscr{L}^{\infty,c}(G, B)$:

$$\|\xi * [\![K]\!]\|_{\mathrm{si}} \leq \||\xi\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(L^2(G,B),\mathcal{E})} \cdot 2\max(\|[\![K]\!]\|_1, \|[\![K]\!]\|_2).$$

Proof. Observe that

$$\|\xi * [\![K]\!]\|_{\mathrm{si}} = \|\xi * [\![K]\!]\|_{\mathcal{E}} + \||\xi * [\![K]\!]\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(L^2(G,B),\mathcal{E})}$$

$$= \||\xi\rangle\!\rangle[\![K^\wedge]\!]\|_{\mathcal{E}} + \||\xi\rangle\!\rangle^{\mathrm{e}} \circ \rho_{[\![K]\!]}\|_{\mathbb{L}^G(L^2(G,B),\mathcal{E})}$$

$$\leq \big\| |\xi\rangle\!\rangle^{\mathrm{e}} \big\|_{\mathbb{L}^G(L^2(G,B),\mathcal{E})} \cdot \big\| [\![K^\wedge]\!] \big\|_2 + \big\| |\xi\rangle\!\rangle^{\mathrm{e}} \big\|_{\mathbb{L}^G(L^2(G,B),\mathcal{E})} \cdot \big\| \rho_{[\![K]\!]} \big\|_{\mathbb{L}^G(L^2(G,B))}$$

$$= \big\| |\xi\rangle\!\rangle^{\mathrm{e}} \big\|_{\mathbb{L}^G(L^2(G,B),\mathcal{E})} \cdot \Big(\big\| [\![K^\wedge]\!] \big\|_2 + \big\| \rho_{[\![K]\!]} \big\|_{\mathbb{L}^G(L^2(G,B))} \Big)$$

$$\leq \big\| |\xi\rangle\!\rangle^{\mathrm{e}} \big\|_{\mathbb{L}^G(L^2(G,B),\mathcal{E})} \cdot \big(\big\| [\![K]\!] \big\|_2 + \big\| [\![K]\!] \big\|_1 \big)$$

$$\leq \big\| |\xi\rangle\!\rangle^{\mathrm{e}} \big\|_{\mathbb{L}^G(L^2(G,B),\mathcal{E})} \cdot 2\max(\big\| [\![K]\!] \big\|_1, \big\| [\![K]\!] \big\|_2). \qquad\qquad \square$$

9. Representations of Twisted Hilbert C^*-Modules

In this section, let $\mathcal{L}$ be a Hilbert (G, B, β, ω)-module and $A \subseteq \mathbb{L}^G(\mathcal{L})$ an essential C^*-subalgebra. By 'essential', we mean that the closed linear span of $A \cdot \mathcal{L} := A[\mathcal{L}]$ is dense in $\mathcal{L}$. By the Cohen Factorization Theorem, we have $A \cdot \mathcal{L} = \mathcal{L}$.

We will be particularly interested in the case $\mathcal{L} = L^2(G, B)$ and $A = C_r^*(G, B, \beta, \omega)$, which is an essential C^*-subalgebra of $\mathbb{L}^G\big(L^2(G, B)\big)$. The presence of the group G is only to ensure that our constructions are invariant with respect to some twisted group action. Note that the results here appear in almost exactly the same form as in Meyer's paper. The only thing to note is that group actions are twisted, and in all cases where a Hilbert-module operator is required to be G-equivariant in some sense, it is meant to be twisted G-equivariant.

Definition 7. An $\mathcal{E}$-concrete Hilbert A-module is a closed linear subspace $\mathcal{F}$ of $\mathbb{L}^G(\mathcal{L}, \mathcal{E})$ that satisfies $\mathcal{F} \circ A \subseteq \mathcal{F}$ and $\mathcal{F}^* \circ \mathcal{F} \subseteq A$. We call $\mathcal{F}$ essential iff the linear span of $\mathcal{F}[\mathcal{L}]$ is dense in $\mathcal{E}$.   ♠

An $\mathcal{E}$-concrete Hilbert A-module $\mathcal{F} \subseteq \mathbb{L}^G(\mathcal{L}, \mathcal{E})$ can be made essential by shrinking $\mathcal{E}$. Indeed, let $\mathcal{E}' \subseteq \mathcal{E}$ denote the closed linear span of $\mathcal{F}[\mathcal{L}]$. Then $\mathcal{E}'$ is a twisted G-invariant Hilbert A-submodule and $\mathcal{F} \subseteq \mathbb{L}^G(\mathcal{L}, \mathcal{E}')$ is an essential $\mathcal{E}'$-concrete Hilbert A-module.

Lemma 5. *An $\mathcal{E}$-concrete Hilbert A-module $\mathcal{F} \subseteq \mathbb{L}^G(\mathcal{L}, \mathcal{E})$ becomes a Hilbert A-module when equipped with the right A-module structure*

$$\forall \xi \in \mathcal{F}, \ \forall a \in A : \quad \xi \cdot a := \xi \circ a$$

and the A-valued inner product

$$\forall \xi, \eta \in \mathcal{F} : \quad \langle \xi | \eta \rangle_{\mathcal{F}} := \xi^* \circ \eta.$$

The Hilbert-module norm and the operator norm on $\mathcal{F}$ then coincide. Furthermore,

$$(9) \qquad\qquad \mathcal{F} = \mathcal{F} \circ A = \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F}$$

and

$$\mathcal{F}[\mathcal{L}] = (\mathcal{F} \circ \mathcal{F}^*)[\mathcal{E}] = (\mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F})[\mathcal{L}].$$

As an immediate consequence, $\mathcal{F}$ is essential iff the linear span of $(\mathcal{F} \circ \mathcal{F}^*)[\mathcal{E}]$ is dense in $\mathcal{E}$.

Proof. We already have by definition that $\xi \cdot a \in \mathcal{F}$ for all $\xi \in \mathcal{F}$ and $a \in A$, and that $\langle \xi | \eta \rangle_{\mathcal{F}} \in A$ for all $\xi, \eta \in \mathcal{F}$. In addition, the conditions

$$\langle \xi | \eta \cdot a \rangle_{\mathcal{F}} = \langle \xi | \eta \rangle_{\mathcal{F}} \cdot a, \quad \langle \xi | \eta \rangle_{\mathcal{F}} = (\langle \eta | \xi \rangle_{\mathcal{F}})^* \quad \text{and} \quad \langle \xi | \xi \rangle_{\mathcal{F}} \geq 0_A$$

for a pre-Hilbert A-module are satisfied. Furthermore, as

$$(10) \qquad\qquad \| \xi \|_{\mathcal{F}} := \sqrt{\langle \xi | \xi \rangle_{\mathcal{F}}} := \sqrt{\| \xi^* \circ \xi \|_A} := \sqrt{\| \xi^* \circ \xi \|_{\mathbb{L}^G(\mathcal{L})}} = \| \xi \|_{\mathbb{L}^G(\mathcal{L})}$$

for all $\xi \in \mathcal{F}$, we find that the Hilbert-module norm and the operator norm coincide. Hence, $\mathcal{F}$ is a Hilbert A-module, and

$$\mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F} \subseteq \mathcal{F} \circ A \subseteq \mathcal{F}.$$

Now, every Hilbert module $\mathcal{M}$ satisfies the property that any $\xi \in \mathcal{M}$ may be written as $\eta \cdot \langle \eta | \eta \rangle_{\mathcal{M}}$ for some $\eta \in \mathcal{M}$. As such, $\mathcal{F} \subseteq \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F}$, which yields

$$\mathcal{F} = \mathcal{F} \circ A = \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F}.$$

Therefore,

$$\mathcal{F}[\mathcal{L}] = (\mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F})[\mathcal{L}] \subseteq (\mathcal{F} \circ \mathcal{F}^*)[\mathcal{E}] \subseteq \mathcal{F}[\mathcal{L}],$$

and so we obtain

$$\mathcal{F}[\mathcal{L}] = (\mathcal{F} \circ \mathcal{F}^*)[\mathcal{E}] = (\mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F})[\mathcal{L}].$$

The final assertion follows immediately from the argument. $\square$

Let $\mathcal{F}$ be a Hilbert A-module. We will construct a canonical representation of $\mathcal{F}$ as an $(\mathcal{F} \bar{\otimes}_A \mathcal{L})$-concrete Hilbert A-module, where $\mathcal{F} \bar{\otimes}_A \mathcal{L}$ is the completed A-balanced tensor product of $\mathcal{F}$ and $\mathcal{L}$. To be more precise, we perform the following steps:

- Form the algebraic A-balanced tensor product $\mathcal{F} \otimes_A \mathcal{L}$.

- Equip it with the (possibly degenerate) A-valued sesquilinear form

(11) $$\langle \xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2 \rangle_{\mathcal{F} \otimes_A \mathcal{L}} := \langle \eta_1 | \langle \xi_1 | \xi_2 \rangle_{\mathcal{F}} \cdot \eta_2 \rangle_{\mathcal{L}}.$$

- Take the quotient of $\mathcal{F} \otimes_A \mathcal{L}$ by the A-linear subspace

$$\mathcal{N} := \{ x \in \mathcal{F} \otimes_A \mathcal{L} \mid \langle x | x \rangle_{\mathcal{F} \otimes_A \mathcal{L}} = 0_A \}$$

   so as to obtain a pre-Hilbert A-module whose A-valued pre-inner product we denote by $\langle \cdot | \cdot \rangle_{\mathcal{F} \bar{\otimes}_A \mathcal{L}}$.

- Complete $(\mathcal{F} \otimes_A \mathcal{L})/\mathcal{N}$ with respect to the norm induced by $\langle \cdot | \cdot \rangle_{\mathcal{F} \bar{\otimes}_A \mathcal{L}}$ to finally get a Hilbert A-module $\mathcal{F} \bar{\otimes}_A \mathcal{L}$.

The construction laid out above is, in fact, part of the standard proof that Morita-Rieffel equivalence is a transitive relation.

Now, equip $\mathcal{F}$ with the trivial action of G, so that $\mathcal{F} \bar{\otimes}_A \mathcal{L}$ becomes a Hilbert (B, G, β, ω)-module. Then the mapping $\mathcal{F} \mapsto \mathcal{F} \bar{\otimes}_A \mathcal{L}$ is functorial. This means that any adjointable operator $\Phi : \mathcal{F}_1 \to \mathcal{F}_2$ between Hilbert A-modules induces a G-equivariant adjointable operator

$$\Phi \bar{\otimes} \mathrm{id}_{\mathcal{L}} : \mathcal{F}_1 \bar{\otimes}_A \mathcal{L} \to \mathcal{F}_2 \bar{\otimes}_A \mathcal{L}$$

between Hilbert (B, G, β, ω)-modules.

Using the isomorphism $A \bar{\otimes}_A \mathcal{L} \cong A \cdot \mathcal{L} = \mathcal{L}$, we thus acquire a mapping

$$T : \mathcal{F} \to \mathbb{L}^G(A \bar{\otimes}_A \mathcal{L}, \mathcal{F} \bar{\otimes}_A \mathcal{L}) \cong \mathbb{L}^G(\mathcal{L}, \mathcal{F} \bar{\otimes}_A \mathcal{L}).$$

More explicitly, we have $(T(\xi))(f) := \xi \otimes f$ and $(T(\xi)^*)(\eta \otimes f) := \langle \xi | \eta \rangle_{\mathcal{F}}(f)$ for all $\xi, \eta \in \mathcal{F}$ and $f \in \mathcal{L}$, where we view $\langle \xi | \eta \rangle_{\mathcal{F}} \in A \subseteq \mathbb{L}^G(\mathcal{L})$.

Theorem 4. *Let $\mathcal{F}$ be a Hilbert A-module and define T as above. Then $T[\mathcal{F}]$ is an essential $(\mathcal{F} \bar{\otimes}_A \mathcal{L})$-concrete Hilbert A-module, and $T : \mathcal{F} \to T[\mathcal{F}]$ is an isomorphism of Hilbert A-modules.*

If $\mathcal{F} \subseteq \mathbb{L}^G(\mathcal{L}, \mathcal{E})$ is already an essential $\mathcal{E}$-concrete Hilbert A-module, then

$$U := \left\{ \begin{array}{ccc} \mathcal{F} \bar{\otimes}_A \mathcal{L} & \to & \mathcal{E} \\ \xi \otimes f & \mapsto & \xi(f) \end{array} \right\}$$

is a G-equivariant unitary mapping that satisfies $U \circ T(\xi) = \xi$ for all $\xi \in \mathcal{F}$. In other words, $\mathcal{F}$ and $T[\mathcal{F}]$ are isomorphic as concrete Hilbert A-modules via U.

Proof. For all $\xi \in \mathcal{F}$ and $a \in A$, we have $T(\xi \cdot a) = T(\xi) \circ a$ because for each $f \in \mathcal{L}$,

$$
\begin{aligned}
(T(\xi \cdot a))(f) &= (\xi \cdot a) \otimes f \\
&= \xi \otimes a \cdot f \qquad \text{(As the tensor product is A-balanced.)} \\
&= \xi \otimes a(f) \\
&= (T(\xi))(a(f)) \\
&= (T(\xi) \circ a)(f).
\end{aligned}
$$

For all $\xi, \eta \in \mathcal{F}$, we have $T(\xi)^* \circ T(\eta) = \langle \xi | \eta \rangle_{\mathcal{F}}$ because for each $f \in \mathcal{L}$,

$$
\big(T(\xi)^* \circ T(\eta)\big)(f) = T(\xi)^*(\eta \otimes f) = \langle \xi | \eta \rangle_{\mathcal{F}}(f).
$$

[(10)](#) shows that T is isometric, so $T[\mathcal{F}]$ must be a closed subset of $\mathbb{L}^G(\mathcal{L}, \mathcal{F} \bar{\otimes}_A \mathcal{L})$. Hence, $T[\mathcal{F}]$ is an $(\mathcal{F} \bar{\otimes}_A \mathcal{L})$-concrete Hilbert A-module, and $T : \mathcal{F} \to T[\mathcal{F}]$ is an isomorphism with respect to the Hilbert A-module structure defined in [Lemma 5](#). Furthermore, $T[\mathcal{F}]$ is essential because $\mathcal{F} \bar{\otimes}_A \mathcal{L}$ is generated by elementary tensors $\xi \otimes f = (T(\xi))(f)$ with $\xi \in \mathcal{F}$ and $f \in \mathcal{L}$.

Now, suppose that $\mathcal{F} \subseteq \mathbb{L}^G(\mathcal{L}, \mathcal{E})$ is an $\mathcal{E}$-concrete Hilbert A-module. By [(11)](#), we get

$$
\begin{aligned}
\| \xi \otimes f \|_{\mathcal{F} \bar{\otimes}_A \mathcal{L}} &= \sqrt{\langle \xi \otimes f | \xi \otimes f \rangle_{\mathcal{F} \bar{\otimes}_A \mathcal{L}}} \\
&= \sqrt{\langle f | \langle \xi | \xi \rangle_A \cdot f \rangle_{\mathcal{L}}} \\
&= \sqrt{\langle f | (\xi^* \circ \xi)(f) \rangle_{\mathcal{L}}} \\
&= \sqrt{\langle f | \xi^*(\xi(f)) \rangle_{\mathcal{L}}} \\
&= \sqrt{\langle \xi(f) | \xi(f) \rangle_{\mathcal{E}}} \\
&= \| \xi(f) \|_{\mathcal{E}},
\end{aligned}
$$

which implies that U is a well-defined isometry. It is also G-equivariant.

If $\mathcal{F}$ is essential, then the range of U is obviously dense so that U is unitary. Finally,

$$
U((T(\xi))(f)) = U(\xi \otimes f) = \xi(f)
$$

for all $\xi \in \mathcal{F}$ and $f \in \mathcal{L}$, i.e., $U \circ T(\xi) = \xi$. $\qquad \square$

Theorem 5. *Let $\mathcal{F} \subseteq \mathbb{L}^G(\mathcal{L}, \mathcal{E})$ be an $\mathcal{E}$-concrete Hilbert A-module. Then the mapping*

$$
|\xi\rangle \circ \langle \eta| \mapsto \xi \circ \eta^* \in \mathcal{F} \circ \mathcal{F}^* \subseteq \mathbb{L}^G(\mathcal{E})
$$

extends to a $*$-isomorphism from $\mathbb{K}(\mathcal{F})$ to the closed linear span of $\mathcal{F} \circ \mathcal{F}^*$ in $\mathbb{L}^G(\mathcal{E})$.

This representation of $\mathbb{K}(\mathcal{F})$ is essential (i.e., the image of $\mathbb{K}(\mathcal{F})$ in $\mathbb{L}^G(\mathcal{E})$ under the $*$-isomorphism is essential) iff $\mathcal{F}$ is essential.

If $\mathcal{F}$ is essential, then we may extend this representation of $\mathbb{K}(\mathcal{F})$ to a strictly continuous and injective unital $*$-homomorphism $\phi : \mathbb{L}(\mathcal{F}) \to \mathbb{L}^G(\mathcal{E})$, whose range is

$$
M := \big\{ x \in \mathbb{L}^G(\mathcal{E}) \mid x \circ \mathcal{F} \subseteq \mathcal{F} \text{ and } x^* \circ \mathcal{F} \subseteq \mathcal{F} \big\}.
$$

Proof. It is clear that M is a C^*-subalgebra of $\mathbb{L}^G(\mathcal{E})$.

Let $D \subseteq \mathbb{L}^G(\mathcal{E})$ be the closed linear span of $\mathcal{F} \circ \mathcal{F}^*$. Then $D^* = D$, and the identities in [(9)](#) yield $D \circ \mathcal{F} \subseteq \mathcal{F}$, so that $D \subseteq M$. Furthermore, for any $x \in M$, we have

$$
\begin{aligned}
(\mathcal{F} \circ \mathcal{F}^*) \circ x &\subseteq \mathcal{F} \circ (\mathcal{F}^* \circ x) = \mathcal{F} \circ (x^* \circ \mathcal{F})^* \subseteq \mathcal{F} \circ \mathcal{F}^* \qquad \text{and} \\
x \circ (\mathcal{F} \circ \mathcal{F}^*) &\subseteq (x \circ \mathcal{F}) \circ \mathcal{F}^* \subseteq \mathcal{F} \circ \mathcal{F}^*,
\end{aligned}
$$

which yield $D \circ x \subseteq D$ and $x \circ D \subseteq D$. Hence, D is a closed two-sided $*$-ideal of M.

Conversely, given $x \in \mathbb{L}^G(\mathcal{E})$, if $x \circ D \subseteq D$ and $D \circ x \subseteq D$, then

$$\begin{aligned}
x \circ \mathcal{F} &= x \circ \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F} \\
&\subseteq x \circ D \circ \mathcal{F} \\
&\subseteq D \circ \mathcal{F} \\
&\subseteq \mathcal{F},
\end{aligned}$$

$$\begin{aligned}
x^* \circ F &= x^* \circ \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F} \\
&\subseteq x^* \circ D \circ \mathcal{F} \\
&= (D^* \circ x)^* \circ \mathcal{F} \\
&= (D \circ x)^* \circ \mathcal{F} \\
&\subseteq D^* \circ \mathcal{F} \\
&= D \circ \mathcal{F} \\
&\subseteq \mathcal{F},
\end{aligned}$$

which yield $x \in M$. Consequently, $x \in M$ iff both $x \circ D \subseteq D$ and $D \circ x \subseteq D$.

Define a $*$-homomorphism $\psi : M \to \mathbb{L}(\mathcal{F})$ by $(\psi(x))(\xi) := x \circ \xi$ for all $x \in M$ and $\xi \in \mathcal{F}$.

- If $\mathcal{F}$ is essential, then ψ is injective: If $x \in M$ and $\psi(x) = 0_{\mathbb{L}(\mathcal{F})}$, then

$$(x \circ \xi)(f) = x(\xi(f)) = 0_{\mathcal{E}}$$

  for all $\xi \in \mathcal{F}$ and $f \in \mathcal{L}$. Therefore, x vanishes on the dense subspace $\mathcal{F}[\mathcal{L}]$ of $\mathcal{E}$, and so $x = 0_{\mathbb{L}^G(\mathcal{E})}$.

- Regardless of whether $\mathcal{F}$ is essential or not, the restriction of ψ to D is injective: If $x \in D$ and $x \circ \mathcal{F} = \{0_{\mathcal{F}}\}$, then

$$x \circ (\mathcal{F} \circ \mathcal{F}^*) = (x \circ \mathcal{F}) \circ \mathcal{F}^* = \left\{0_{\mathbb{L}^G(\mathcal{E})}\right\},$$

  so that $x \circ D = \left\{0_{\mathbb{L}^G(\mathcal{E})}\right\}$. Therefore,

$$x \circ x^* \in x \circ D^* = x \circ D = \left\{0_{\mathbb{L}^G(\mathcal{E})}\right\},$$

  which means that $x = 0_{\mathbb{L}^G(\mathcal{E})}$.

As $\psi(\xi \circ \eta^*) = |\xi\rangle \circ \langle\eta|$ for all $\xi, \eta \in \mathcal{F}$, and as $\psi|_D : D \to \mathbb{L}(\mathcal{F})$ is an injective, hence isometric, $*$-homomorphism, we immediately get the following facts:

- $\psi[D] = \mathbb{K}(\mathcal{F})$.

- $(\psi|_D)^{\leftarrow} : \mathbb{K}(\mathcal{F}) \to D$ equals the linear extension of the mapping $\left\{\begin{array}{ccc} |\mathcal{F}\rangle \circ \langle\mathcal{F}| & \to & D \\ |\xi\rangle \circ \langle\eta| & \mapsto & \xi \circ \eta^* \end{array}\right\}$.

Now that we have

$$(\psi|_D)^{\leftarrow}[\mathbb{K}(\mathcal{F})] = D := \overline{\mathrm{Span}(\mathcal{F} \circ \mathcal{F}^*)},$$

Lemma 5 implies that $(\psi|_D)^{\leftarrow} : \mathbb{K}(\mathcal{F}) \to D \subseteq \mathbb{L}^G(\mathcal{E})$ is an essential $*$-representation of $\mathbb{K}(\mathcal{F})$ in $\mathbb{L}^G(\mathcal{E})$ iff $\mathcal{F}$ is essential.

Suppose that $\mathcal{F}$ is essential; it is C^*-folklore that $(\psi|_D)^{\leftarrow}$ can be uniquely extended to a strictly continuous and *injective* unital $*$-homomorphism ϕ from $\mathbb{L}(\mathcal{F}) \cong M(\mathbb{K}(\mathcal{F}))$ to $\mathbb{L}^G(\mathcal{E})$. Then as $\mathbb{K}(\mathcal{F})$ is an ideal of $\mathbb{L}(\mathcal{F})$, we obtain

$$\begin{aligned}
\phi(T) \circ \mathcal{F} &= \phi(T) \circ D \circ \mathcal{F} \quad &&\text{(As } D \circ \mathcal{F} = \mathcal{F} \text{ by (9).)} \\
&= \phi(T) \circ \phi[\mathbb{K}(\mathcal{F})] \circ \mathcal{F} \quad &&\text{(As } \phi[\mathbb{K}(\mathcal{F})] = (\psi|_D)^{\leftarrow}[\mathbb{K}(\mathcal{F})] = D.) \\
&= \phi[T \circ \mathbb{K}(\mathcal{F})] \circ \mathcal{F} \\
&\subseteq \phi[\mathbb{K}(\mathcal{F})] \circ \mathcal{F}
\end{aligned}$$

$$= D \circ \mathcal{F}$$
$$= \mathcal{F}$$

for all $T \in \mathbb{L}(\mathcal{F})$, and so $\phi(T)^* \circ \mathcal{F} = \phi(T^*) \circ \mathcal{F} \subseteq \mathcal{F}$ for all $T \in \mathbb{L}(\mathcal{F})$ as well. Hence, the range of ϕ is contained in M.

To show that the range of ϕ is precisely M, it suffices to establish that $\phi \circ \psi = \mathrm{id}_M$. We begin by picking $x \in M$. Then for all $K \in \mathbb{K}(\mathcal{F})$ and $\eta \in \mathcal{E}$, we have

$$
\begin{aligned}
((\phi \circ \psi)(x))((\phi(K))(\eta)) &= (\phi(\psi(x)))((\phi(K))(\eta)) \\
&= (\phi(\psi(x)) \circ \phi(K))(\eta) \\
&= (\phi(\psi(x) \circ K))(\eta) \\
&= (\phi(\psi(x) \circ \psi(\phi(K))))(\eta) \\
&= (\phi(\psi(x \circ \phi(K))))(\eta) \\
&= (x \circ \phi(K))(\eta) \\
&= x((\phi(K))(\eta)).
\end{aligned}
$$

Hence, $(\phi \circ \psi)(x)$ and x coincide on the dense subspace $\phi(K)[\mathcal{E}] \subseteq E$, and so $(\phi \circ \psi)(x) = x$. As $x \in M$ is arbitrary, we conclude that $\phi \circ \psi = \mathrm{id}_M$. This finishes the proof. $\qquad\square$

10. Continuously Square-Integrable Twisted Hilbert Modules

In this section, we maintain the abbreviations $\mathcal{L} = L^2(G, B)$ and $A = C_r^*(G, B, \beta, \omega)$.

Definition 8. A subset $\mathcal{R}$ of $\mathcal{E}_{\mathrm{si}}$ is called relatively continuous iff

$$\langle\!\langle \mathcal{R} | \mathcal{R} \rangle\!\rangle^{\mathrm{e}} := \langle\!\langle \mathcal{R} | \circ | \mathcal{R} \rangle\!\rangle^{\mathrm{e}} \subseteq C_r^*(G, B, \beta, \omega).$$

Note: The phrase 'relatively continuous' was first coined by Ruy Exel in [4]. $\qquad\spadesuit$

Suppose that $\mathcal{R}$ is a relatively continuous subset of $\mathcal{E}$. Following Meyer, define

$$\mathcal{F} := \boldsymbol{\mathcal{F}}(\mathcal{E}, \mathcal{R}) \subseteq \mathbb{L}^G\big(L^2(G, B), \mathcal{E}\big)$$

to be the closed linear span of $|\mathcal{R}\rangle\!\rangle^{\mathrm{e}} \cup (|\mathcal{R}\rangle\!\rangle^{\mathrm{e}} \circ C_r^*(G, B, \beta, \omega))$. Then

$$\mathcal{F} \circ C_r^*(G, B, \beta, \omega) \subseteq \mathcal{F} \quad \text{and} \quad \mathcal{F}^* \circ \mathcal{F} \subseteq C_r^*(G, B, \beta, \omega).$$

It thus follows from Lemma 5 that $\mathcal{F}$ has the structure of a Hilbert $C_r^*(G, B, \beta, \omega)$-module.

The closed linear span of $\mathcal{F} \circ \mathcal{F}^* \subseteq \mathbb{L}^G(\mathcal{E})$ is called the generalized fixed-point algebra of $\mathcal{E}$ with respect to $\mathcal{R}$, and we will denote it by $\mathrm{Fix}(\mathcal{E}, \mathcal{R})$.

We will show that there is a canonical isomorphism between the C^*-algebras $\mathrm{Fix}(\mathcal{E}, \mathcal{R})$ and $\mathbb{K}(\mathcal{F})$, the latter being the algebra of compact operators on $\mathcal{F}$. Hence, $\mathrm{Fix}(\mathcal{E}, \mathcal{R})$ is Morita-Rieffel equivalent to the closed ideal J of $C_r^*(G, B, \beta, \omega)$ generated by $\mathcal{F}^* \circ \mathcal{F}$. This last point is easily justified as follows. Observe that $\mathcal{F}$, being a Hilbert $C_r^*(G, B, \beta, \omega)$-module, is automatically a Hilbert J-module. Then as $\langle \mathcal{F} | \mathcal{F} \rangle_{\mathcal{F}} = \mathcal{F}^* \circ \mathcal{F}$ is a dense linear subspace of J, it follows that $\mathcal{F}$ is a *full* Hilbert J-module. By a standard result in the theory of Morita-Rieffel equivalence, J is therefore Morita-Rieffel equivalent to $\mathbb{K}(\mathcal{F})$.

Definition 9. We call $\mathcal{E}$ proper iff $\mathcal{E}$ contains a dense relatively continuous subset. $\qquad\spadesuit$

This definition of properness is consistent with the one given by Rieffel in [8]. In Section 7 of [9], Rieffel avoids projective group representations by using a central extension to produce a unitary group representation. This is a rather standard procedure and is described in [10]. Our framework

deals directly with projective representations, but the small price to pay, in order to avoid measure-theoretical issues, is to work with second-countable locally compact Hausdorff groups. This, however, is not really a disadvantage because the locally compact Hausdorff groups that we mostly encounter are of the kind just mentioned.

There are some issues mentioned in [5] that concern relatively continuous subsets. Firstly, the square-integrability of $\mathcal{E}$ does not necessarily guarantee the existence of a dense relatively continuous subset. Secondly, even if such dense subsets exist, our construction of $\mathcal{F}(\mathcal{E}, \mathcal{R})$ and $\mathrm{Fix}(\mathcal{E}, \mathcal{R})$ might depend on the choice of $\mathcal{R}$; this might lead to a multitude of Hilbert $C_r^*(G, B, \beta, \omega)$-modules playing the role of an imprimitivity bi-module, thus preventing $\mathcal{F}$ and Fix from being canonical constructs, i.e., depending on nothing else other than the structure of $\mathcal{E}$.

We can sweep the first issue under the rug by simply working only with twisted Hilbert modules that possess dense relatively continuous subsets. However, we would then have to deal with the second issue by postulating some condition(s) that would at least tell us how to pick a relatively continuous subset $\mathcal{R}$ in a canonical fashion. We will concern ourselves with the solution to this problem in the sections to follow, based on Meyer's ideas.

In what follows, we define Condition (S) to be the following statement:

Condition (S): The mappings $\beta : B \to \mathrm{Aut}(\mathcal{E})$ and $\gamma^{\mathcal{E}} : G \to \mathbb{U}(\mathcal{E})$ are strongly continuous at the identity element e of G, and $\omega : G \times G \to M(B)$ is strictly continuous on $(G \times \{e\}) \cup (\{e\} \times G)$.

Proposition 5. *Let $\mathcal{R}$ be a relative continuous subset of $\mathcal{E}$. Then $\mathcal{F}(\mathcal{E}, \mathcal{R})$ is an $\mathcal{E}$-concrete Hilbert A-module. If Condition (S) holds and $\mathcal{R}$ is a dense subset of $\mathcal{E}$, then $\mathcal{F}(\mathcal{E}, \mathcal{R})$ is essential.*

Proof. By construction, $\mathcal{F} := \mathcal{F}(\mathcal{E}, \mathcal{R})$ is a closed linear subspace of $\mathbb{L}^G(\mathcal{L}, \mathcal{E})$ and $\mathcal{F} \circ A \subseteq \mathcal{F}$. Letting $\mathcal{R}$ be a relatively continuous subset of $\mathcal{E}$, we also have $\mathcal{F}^* \circ \mathcal{F} \subseteq A$. Therefore, $\mathcal{F}$ is an $\mathcal{E}$-concrete Hilbert A-module.

For all $\xi \in \mathcal{E}$ and $f \in \mathscr{L}^{\infty,c}(G, B)$, we have the inequality

$$\|\xi * f\|_{\mathcal{E}} = \||\xi\rangle\!\rangle[\![f^\wedge]\!]\|_{\mathcal{E}} = \left\| \int_G \gamma_x(\xi) \cdot f^\wedge(x) \, \mathrm{d}x \right\|_{\mathcal{E}} \leq \|\xi\|_{\mathcal{E}} \|[\![f^\wedge]\!]\|_1.$$

Suppose that Condition (S) holds. Then $\mathcal{E} * L^{\infty,c}(G, B)$ is a dense subset of $\mathcal{E}$. Let $\mathcal{R}$ be a dense subset of $\mathcal{E}$. It then follows from the inequality that $\mathcal{R} * L^{\infty,c}(G, B)$ is a dense subset of $\mathcal{E}$ as well. Finally, as

$$\begin{aligned}
\mathcal{R} * L^{\infty,c}(G, B) &= |\mathcal{R}\rangle\!\rangle[L^{\infty,c}(G, B)^\vee] \\
&= |\mathcal{R}\rangle\!\rangle[L^{\infty,c}(G, B)] \\
&= |\mathcal{R}\rangle\!\rangle^{\mathrm{e}}[L^{\infty,c}(G, B)] \qquad (\text{As } \mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}.) \\
&\subseteq \mathcal{F}[\mathcal{L}],
\end{aligned}$$

we conclude that $\mathcal{F}[\mathcal{L}]$ is a dense subset of $\mathcal{E}$. $\square$

Proposition 6. *Let $\mathcal{F} \subseteq \mathbb{L}^G(\mathcal{L}, \mathcal{E})$ be an $\mathcal{E}$-concrete Hilbert A-module, and define*

$$\boldsymbol{\mathcal{R}}_{\mathcal{F}} := \{\zeta \in \mathcal{E}_{\mathrm{si}} \mid |\zeta\rangle\!\rangle^{\mathrm{e}} \in \mathcal{F}\};$$

$$\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0 := \{\xi([\![K]\!]) \in \mathcal{E} \mid \xi \in \mathcal{F} \text{ and } K \in \mathscr{L}^{\infty,c}(G, B)\}.$$

Then the following statements hold:

- $\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0 \subseteq \boldsymbol{\mathcal{R}}_{\mathcal{F}}$.

- *Both $\boldsymbol{\mathcal{R}}_{\mathcal{F}}$ and $\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0$ are relatively continuous subsets of $\mathcal{E}$.*

- *Both $|\boldsymbol{\mathcal{R}}_{\mathcal{F}}\rangle\!\rangle^{\mathrm{e}}$ and $|\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0\rangle\!\rangle^{\mathrm{e}}$ are dense subsets of $\mathcal{F}$.*

- $\boldsymbol{\mathcal{F}}\big(\mathcal{E}, \boldsymbol{\mathcal{R}}_{\mathcal{F}}^0\big) = \boldsymbol{\mathcal{F}}(\mathcal{E}, \boldsymbol{\mathcal{R}}_{\mathcal{F}}) = \mathcal{F}$.

Proof. It is already clear by definition that $\boldsymbol{\mathcal{R}}_{\mathcal{F}}$ is a relatively continuous subset of $\mathcal{E}$.

Let $\xi \in \mathcal{F}$ and $K \in \mathscr{L}^{\infty,c}(G, B)$. Then
$$\big\langle \gamma_g^{\mathcal{E}}(\xi(\llbracket K \rrbracket)) \big| \eta \big\rangle_{\mathcal{E}} = \big\langle \xi\big(\gamma_g^{\mathcal{L}}(\llbracket K \rrbracket)\big) \big| \eta \big\rangle_{\mathcal{E}} \qquad \text{(As ξ is G-equivariant.)}$$
$$= \big\langle \gamma_g^{\mathcal{L}}(\llbracket K \rrbracket) \big| \xi^*(\eta) \big\rangle_{\mathcal{L}} \qquad \text{(As ξ is adjointable.)}$$
for all $g \in G$ and $\eta \in \mathcal{E}$. As $L^{\infty,c}(G, B) \subseteq L^2(G, B)_{\mathrm{si}}$, we thus obtain $\xi(\llbracket K \rrbracket) \in \mathcal{E}_{\mathrm{si}}$. Identities from earlier on now yield
$$|\xi(\llbracket K \rrbracket)\rangle\!\rangle^{\mathrm{e}} = \xi \circ |\llbracket K \rrbracket\rangle\!\rangle^{\mathrm{e}} = \xi \circ \rho_{\llbracket K^\vee \rrbracket} \subseteq \mathcal{F} \cdot A \subseteq \mathcal{F},$$
so we get $\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0 \subseteq \boldsymbol{\mathcal{R}}_{\mathcal{F}}$. Therefore, $\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0$ a relatively continuous subset of $\mathcal{E}$.

The computation above also shows that
$$|\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0\rangle\!\rangle^{\mathrm{e}} = \mathcal{F} \cdot \text{Image of } L^{\infty,c}(G, B) \text{ under } \rho.$$
As the image of $L^{\infty,c}(G, B)$ under ρ is dense in A, and as the right action of A on $\mathcal{F}$ is non-degenerate, we can deduce rightaway that both $|\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0\rangle\!\rangle^{\mathrm{e}}$ and $|\boldsymbol{\mathcal{R}}_{\mathcal{F}}\rangle\!\rangle^{\mathrm{e}}$ are dense subsets of $\mathcal{F}$.

According to definition,
$$|\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0\rangle\!\rangle^{\mathrm{e}} \subseteq \boldsymbol{\mathcal{F}}\big(\mathcal{E}, \boldsymbol{\mathcal{R}}_{\mathcal{F}}^0\big) = \overline{\mathrm{Span}\big(|\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0\rangle\!\rangle^{\mathrm{e}} \cup (|\boldsymbol{\mathcal{R}}_{\mathcal{F}}^0\rangle\!\rangle^{\mathrm{e}} \circ A)\big)} \subseteq \overline{\mathcal{F}} = \mathcal{F} \quad \text{and}$$
$$|\boldsymbol{\mathcal{R}}_{\mathcal{F}}\rangle\!\rangle^{\mathrm{e}} \subseteq \boldsymbol{\mathcal{F}}(\mathcal{E}, \boldsymbol{\mathcal{R}}_{\mathcal{F}}) = \overline{\mathrm{Span}(|\boldsymbol{\mathcal{R}}_{\mathcal{F}}\rangle\!\rangle^{\mathrm{e}} \cup (|\boldsymbol{\mathcal{R}}_{\mathcal{F}}\rangle\!\rangle^{\mathrm{e}} \circ A))} \subseteq \overline{\mathcal{F}} = \mathcal{F}.$$
Therefore, $\boldsymbol{\mathcal{F}}\big(\mathcal{E}, \boldsymbol{\mathcal{R}}_{\mathcal{F}}^0\big) = \boldsymbol{\mathcal{F}}(\mathcal{E}, \boldsymbol{\mathcal{R}}_{\mathcal{F}}) = \mathcal{F}$. $\qquad\qquad\square$

Definition 10. We call a linear subspace $\mathcal{R}$ of $\mathcal{E}$ complete iff $\mathcal{R}$ is a linear subspace of $\mathcal{E}_{\mathrm{si}}$ that satisfies $\mathcal{R} * L^{\infty,c}(G, B) \subseteq \mathcal{R}$ and is complete with respect to the norm $\|\cdot\|_{\mathrm{si}}$ defined in (1). If $\mathcal{R}$ is a subset of $\mathcal{E}_{\mathrm{si}}$, then we define its completion to be the smallest complete linear subspace of $\mathcal{E}_{\mathrm{si}}$ containing $\mathcal{R}$, i.e., the $\|\cdot\|_{\mathrm{si}}$-closed linear span of $\mathcal{R} \cup (\mathcal{R} * L^{\infty,c}(G, B))$. $\qquad\spadesuit$

Definition 11. A continuously square-integrable Hilbert (G, B, β, ω)-module is an ordered pair $(\mathcal{E}, \mathcal{R})$, where $\mathcal{E}$ is a Hilbert (G, B, β, ω)-module and $\mathcal{R}$ is a complete and dense relatively continuous linear subspace of $\mathcal{E}$. $\qquad\spadesuit$

We will now prove, under Condition (S), that there exists a one-to-one correspondence between isomorphism classes of Hilbert A-modules and isomorphism classes of continuously square-integrable Hilbert (G, B, β, ω)-modules.

Theorem 6. *Assume Condition (S).*

The mapping $\mathcal{F} \mapsto \boldsymbol{\mathcal{R}}_{\mathcal{F}}$ is a bijection from the set of $\mathcal{E}$-concrete Hilbert A-modules $\mathcal{F} \subseteq \mathbb{L}^G(\mathcal{L}, \mathcal{E})$ onto the set of complete relatively continuous linear subspaces of $\mathcal{E}$. Its inverse is $\mathcal{R} \mapsto \boldsymbol{\mathcal{F}}(\mathcal{E}, \mathcal{R})$.

An $\mathcal{E}$-concrete Hilbert A-module $\mathcal{F}$ is essential iff $\boldsymbol{\mathcal{R}}_{\mathcal{F}}$ is dense.

Furthermore, isomorphism classes of Hilbert A-modules correspond bijectively to isomorphism classes of continuously square-integrable Hilbert (G, B, β, ω)-modules.

Proof. Let $\mathcal{F}$ be an $\mathcal{E}$-concrete Hilbert A-module. We already know that $\boldsymbol{\mathcal{R}}_{\mathcal{F}}$ is a relatively continuous subset of $\mathcal{E}$, and by the argument used in the proof of Proposition 4, its completeness can be shown. Furthermore, Proposition 6 asserts that $\boldsymbol{\mathcal{F}}(\mathcal{E}, \boldsymbol{\mathcal{R}}_{\mathcal{F}}) = \mathcal{F}$.

Conversely, suppose that $\mathcal{R}$ is a complete and relatively continuous subset of $\mathcal{E}$, and let $\mathcal{F} := \boldsymbol{\mathcal{F}}(\mathcal{E}, \mathcal{R})$. Then $\mathcal{R} \subseteq \boldsymbol{\mathcal{R}}_{\mathcal{F}}$, and our claim is that $\mathcal{R} = \boldsymbol{\mathcal{R}}_{\mathcal{F}}$.

Let $\zeta \in \mathcal{R}_\mathcal{F}$. By assumption of the completeness of $\mathcal{R}$, we have $\mathcal{R} * L^{\infty,c}(G, B) \subseteq \mathcal{R}$. Then

$$|\mathcal{R}\rangle\!\rangle^e \cup (|\mathcal{R}\rangle\!\rangle^e \circ \rho[L^{\infty,c}(G,B)]) = |\mathcal{R}\rangle\!\rangle^e \cup (|\mathcal{R} * L^{\infty,c}(G,B)\rangle\!\rangle^e) \subseteq |\mathcal{R}\rangle\!\rangle^e \subseteq \mathcal{F}.$$

As $\mathcal{F} = \boldsymbol{\mathcal{F}}(\mathcal{E}, \mathcal{R})$, it is the closure of $|\mathcal{R}\rangle\!\rangle^e \cup (|\mathcal{R}\rangle\!\rangle^e \circ \rho[L^{\infty,c}(G,B)])$ in $\mathbb{L}^G(\mathcal{L}, \mathcal{E})$. It follows readily that $\mathcal{F}$ is also the closure of $|\mathcal{R}\rangle\!\rangle^e$ in $\mathbb{L}^G(\mathcal{L}, \mathcal{E})$, so we can find a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$ such that $\lim_{n\to\infty} |\zeta_n\rangle\!\rangle^e = |\zeta\rangle\!\rangle^e$ with respect to the norm $\|\cdot\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})}$.

Let $(e_i)_{i\in I}$ be an approximate delta contained in $L^{\infty,c}(G, B)$. By Lemma 4,

$$\lim_{n\to\infty} \|\zeta_n * e_i - \zeta * e_i\|_{\mathrm{si}} = 0$$

for each $i \in I$. Hence, $\zeta * e_i \in \mathcal{R}$ because $\mathcal{R}$ is complete. Then as $(\rho_{e_i})_{i\in I}$ is an approximate identity for A, we have $\xi \cdot \rho_{e_i} \to \xi$ for all elements ξ of a Hilbert A-module; in particular, $|\zeta * e_i\rangle\!\rangle^e = |\zeta\rangle\!\rangle^e \circ \rho_{e_i}$ converges to $|\zeta\rangle\!\rangle^e$ in $\mathbb{L}^G(\mathcal{L}, \mathcal{E})$. This means that $\zeta * e_i \to \zeta$ with respect to $\|\cdot\|_{\mathrm{si}}$, because Condition (S) guarantees that $\zeta * e_i \to \zeta$ in $\mathcal{E}$. Therefore, $\zeta \in \mathcal{R}$, which proves that $\mathcal{R} = \mathcal{R}_\mathcal{F}$.

If $\mathcal{F}$ is essential, then $\boldsymbol{\mathcal{R}}_\mathcal{F}^0$ is dense in $\mathcal{E}$, which makes $\boldsymbol{\mathcal{R}}_\mathcal{F}$ dense in $\mathcal{E}$. Conversely, if $\boldsymbol{\mathcal{R}}_\mathcal{F}$ is dense in $\mathcal{E}$, then $\mathcal{F}$ is essential by Proposition 5.

The last assertion of the theorem follows from Theorem 4. $\qquad\square$

The following proposition shows that the completion of a relatively continuous subset of $\mathcal{E}$ is still relatively continuous. This allows us to consider only complete and relatively continuous linear subspaces of $\mathcal{E}$ without any loss of generality.

Proposition 7. *Let $\mathcal{R}$ be a relatively continuous subset of $\mathcal{E}$. Then $\boldsymbol{\mathcal{R}}_{\boldsymbol{\mathcal{F}}(\mathcal{E},\mathcal{R})}$ is the completion of $\mathcal{R}$, which means that the completion of $\mathcal{R}$ is still relatively continuous.*

Proof. By the previous theorem, $\boldsymbol{\mathcal{R}}_{\boldsymbol{\mathcal{F}}(\mathcal{E},\mathcal{R})}$ is a complete and relatively continuous linear subspace of $\mathcal{E}$. It is also clear by definition that $\mathcal{R} \subseteq \boldsymbol{\mathcal{R}}_{\boldsymbol{\mathcal{F}}(\mathcal{E},\mathcal{R})}$.

Now, let $\mathcal{R}'$ be a complete and relatively continuous linear subspace of $\mathcal{E}$ containing $\mathcal{R}$. We know that $\mathcal{R}' = \boldsymbol{\mathcal{R}}_{\boldsymbol{\mathcal{F}}(\mathcal{E},\mathcal{R}')}$. As $\mathcal{R} \subseteq \mathcal{R}'$, we have $\boldsymbol{\mathcal{F}}(\mathcal{E}, \mathcal{R}) \subseteq \boldsymbol{\mathcal{F}}(\mathcal{E}, \mathcal{R}')$, which in turn yields

$$\boldsymbol{\mathcal{R}}_{\boldsymbol{\mathcal{F}}(\mathcal{E},\mathcal{R})} \subseteq \boldsymbol{\mathcal{R}}_{\boldsymbol{\mathcal{F}}(\mathcal{E},\mathcal{R}')} = \mathcal{R}'.$$

Therefore, $\boldsymbol{\mathcal{R}}_{\boldsymbol{\mathcal{F}}(\mathcal{E},\mathcal{R})}$ is contained in every complete and relatively continuous linear subspace of $\mathcal{E}$, so it is the completion of $\mathcal{R}$. $\qquad\square$

Proposition 8. *Let $\mathcal{R} \subseteq \mathcal{E}$ be a relatively continuous linear subspace. Equip $\mathcal{R}$ with the norm $\|\cdot\|_{\mathrm{si}}$. Then $\mathcal{R}$ is G-invariant. Furthermore, $\mathcal{R}$ is an essential right B-module, i.e., $\mathcal{R} \cdot B = \mathcal{R}$.*

Proof. According to Theorem 6, there exists a concrete Hilbert A-module $\mathcal{F}$ such that $\mathcal{R} = \boldsymbol{\mathcal{R}}_\mathcal{F}$. Let $x \in \mathcal{R}$. Then $|x\rangle\!\rangle^e \cdot a \in \mathcal{F} \cdot A \subseteq \mathcal{F}$, and Cohen's Factorization Theorem implies that

$$\left\{ \begin{array}{ccc} A & \to & \mathcal{F} \\ a & \mapsto & |x\rangle\!\rangle^e \circ a \end{array} \right\}$$

extends to a linear operator from $M(A)$ to $\mathcal{F}$ that is continuous with respect to the strict topology on $M(A)$ and the norm topology on $\mathcal{F}$. Furthermore, if $y \in \mathcal{E}_{\mathrm{si}}$ and $|y\rangle\!\rangle^e = |x\rangle\!\rangle^e \cdot a \in \mathcal{F}$ for some $a \in A$, then by the definition of $\boldsymbol{\mathcal{R}}_\mathcal{F}$, we have $y \in \boldsymbol{\mathcal{R}}_\mathcal{F}$.

Let $b \in B$ and $g \in G$. As $|x \cdot b\rangle\!\rangle^e = |x\rangle\!\rangle^e \circ \rho_b$ and $|\gamma_g(x)\rangle\!\rangle^e = |x\rangle\!\rangle^e \circ \rho_g^*$, and as ρ_b and ρ_g can be viewed as elements of $M(A)$, we immediately get $x \cdot b \in \mathcal{R}$ and $\gamma_g(x) \in \mathcal{R}$ from the previous paragraph.

Let $(u_i)_{i\in I}$ be an approximate identity of B. Then

$$\lim_{i\in I} \|x \cdot u_i - x\|_{\mathrm{si}} = \lim_{i\in I} \|x \cdot u_i - x\|_{\mathcal{E}} + \lim_{i\in I} \||x \cdot u_i\rangle\!\rangle^e - |x\rangle\!\rangle^e\|_{\mathbb{L}(L^2(G,B),\mathcal{E})}$$

$$= \lim_{i \in I} \|x \cdot u_i - x\|_{\mathcal{E}} + \lim_{i \in I} \| |x\rangle\!\rangle^e \circ \rho_{u_i} - |x\rangle\!\rangle^e \|_{\mathbb{L}(L^2(G,B),\mathcal{E})}$$
$$= 0,$$

or equivalently, $x \cdot u_i \to x$ with respect to $\|\cdot\|_{\mathrm{si}}$. The reason for the last equality is as follows. Clearly, ρ_{u_i} converges strictly to $1_{M(A)}$, and as $a \mapsto |x\rangle\!\rangle^e \cdot a$ has a continuous extension to the domain $M(A)$ that is continuous with respect to the strict topology on $M(A)$ and the norm topology on $\mathcal{F}$, it follows that $|x\rangle\!\rangle^e \circ \rho_{u_i}$ converges in $\mathcal{F}$ to $|x\rangle\!\rangle^e$. Hence, $\lim_{i \in I} \| |x\rangle\!\rangle^e \circ \rho_{u_i} - |x\rangle\!\rangle^e \|_{\mathbb{L}(L^2(G,B),\mathcal{E})} = 0$.

By Cohen's Factorization Theorem once more, we therefore conclude that $\mathcal{R} \cdot B = \mathcal{R}$. $\qquad \square$

Definition 12. Let $(\mathcal{E}, \mathcal{R})$ and $(\mathcal{E}', \mathcal{R}')$ be continuously square-integrable Hilbert (G, B, β, ω)-modules. An operator $T \in \mathbb{L}^G(\mathcal{E}, \mathcal{E}')$ is called $(\mathcal{R}, \mathcal{R}')$-continuous iff $T[\mathcal{R}] \subseteq \mathcal{R}'$ and $T^*[\mathcal{R}'] \subseteq \mathcal{R}$. $\spadesuit$

We now come to the main classification theorem.

Theorem 7. *Let $(\mathcal{E}, \mathcal{R})$ be a continuously square-integrable Hilbert (G, B, β, ω)-module, and let $\mathcal{F} := \boldsymbol{\mathcal{F}}(\mathcal{E}, \mathcal{R})$. Then there exists a canonical, injective and strictly continuous $*$-homomorphism $\phi : \mathbb{L}(\mathcal{F}) \to \mathbb{L}^G(\mathcal{E})$ that maps $\mathbb{K}(\mathcal{F})$ isometrically onto $\mathrm{Fix}(\mathcal{E}, \mathcal{R})$.*

$\mathrm{Fix}(\mathcal{E}, \mathcal{R})$ is Morita-Rieffel equivalent to an ideal of $C_r^*(G, B, \beta, \omega)$, namely, the closed linear span of $\langle\!\langle \mathcal{R} | \mathcal{R} \rangle\!\rangle^e \in C_r^*(G, B, \beta, \omega)$.

Under Condition (S), the range of ϕ is the space of $(\mathcal{R}, \mathcal{R})$-continuous operators.

Proof. As $\mathcal{R} * L^{\infty,c}(G, B) \subseteq \mathcal{R}$ by the assumption of completeness on $\mathcal{R}$, we have
$$|\mathcal{R}\rangle\!\rangle^e \circ \rho[L^{\infty,c}(G,B)] = |\mathcal{R} * L^{\infty,c}(G,B)\rangle\!\rangle^e \subseteq |\mathcal{R}\rangle\!\rangle^e \subseteq \mathcal{F}.$$
Then
$$|\mathcal{R}\rangle\!\rangle^e \cup (|\mathcal{R}\rangle\!\rangle^e \circ \rho[L^{\infty,c}(G,B)]) \subseteq |\mathcal{R}\rangle\!\rangle^e \subseteq \mathcal{F},$$
and as
$$\overline{\mathrm{Span}(|\mathcal{R}\rangle\!\rangle^e \cup (|\mathcal{R}\rangle\!\rangle^e \circ \rho[L^{\infty,c}(G,B)]))}^{\mathbb{L}^G(\mathcal{L},\mathcal{E})} = \overline{\mathrm{Span}(|\mathcal{R}\rangle\!\rangle^e \cup (|\mathcal{R}\rangle\!\rangle^e \circ C_r^*(G,B,\beta,\omega)))}^{\mathbb{L}^G(\mathcal{L},\mathcal{E})} = \mathcal{F},$$
we see that $|\mathcal{R}\rangle\!\rangle^e$ is dense in $\mathcal{F}$. Therefore, $\langle\!\langle \mathcal{R} | \mathcal{R} \rangle\!\rangle^e$ is dense in $\mathcal{F}^* \circ \mathcal{F}$ and $|\mathcal{R}\rangle\!\rangle^e \langle\!\langle \mathcal{R}|$ is dense in $\mathcal{F} \circ \mathcal{F}^*$.

The assertions of the first paragraph follow from Theorem 5 if we define ϕ as over there.

As $\mathcal{F} \circ A \subseteq \mathcal{F}$, the closed linear span of J of $\mathcal{F}^* \circ \mathcal{F}$ is a closed ideal of A. Then as mentioned earlier on, $\mathcal{F}$ is an imprimitivity $(\mathbb{K}(\mathcal{F}), J)$-bimodule. Therefore, as $\phi|_{\mathbb{K}(\mathcal{F})} : \mathbb{K}(\mathcal{F}) \to \mathrm{Fix}(\mathcal{E}, \mathcal{R})$ is a $*$-isomorphism, we conclude that $\mathrm{Fix}(\mathcal{E}, \mathcal{R})$ and J are Morita-Rieffel equivalent.

We will now use (3) to show that the space M defined in Theorem 5 is the space of $(\mathcal{R}, \mathcal{R})$-continuous operators.

Claim 1: Every $(\mathcal{R}, \mathcal{R})$-continuous operator is an element of M.

Proof of Claim 1. Let x be an $(\mathcal{R}, \mathcal{R})$-continuous operator. Then $x[\mathcal{R}] \subseteq \mathcal{R}$ and $x^*[\mathcal{R}] \subseteq \mathcal{R}$. Using (3), we have
$$x \circ |\mathcal{R}\rangle\!\rangle^e = |x[\mathcal{R}]\rangle\!\rangle^e \subseteq |\mathcal{R}\rangle\!\rangle^e,$$
which yields
$$x \circ \mathcal{F} = x \circ \overline{|\mathcal{R}\rangle\!\rangle^e}^{\mathbb{L}^G(\mathcal{L},\mathcal{E})} \subseteq \overline{x \circ |\mathcal{R}\rangle\!\rangle^e}^{\mathbb{L}^G(\mathcal{L},\mathcal{E})} \subseteq \overline{|\mathcal{R}\rangle\!\rangle^e}^{\mathbb{L}^G(\mathcal{L},\mathcal{E})} = \mathcal{F}.$$
Similarly, we have $x^* \circ \mathcal{F} \subseteq \mathcal{F}$. Therefore, $x \in M$. $\qquad \square$

Claim 2: Every element of M is an $(\mathcal{R}, \mathcal{R})$-continuous operator.

Proof of Claim 2. The proof of this is more complicated as the argument involves a tight interplay between the norms $\|\cdot\|_{\mathcal{E}}$ and $\|\cdot\|_{\mathcal{E}_{\mathrm{si}}}$. We begin by letting $x \in M$.

First, we show that $\mathcal{F}[L^{\infty,c}(G,B)] \subseteq \mathcal{R}$. Let $T \in \mathcal{F}$ and $K \in \mathscr{L}^{\infty,c}(G,B)$. As $|\mathcal{R}\rangle\!\rangle^{\mathrm{e}}$ is dense in $\mathcal{F}$ with respect to $\|\cdot\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})}$, we can find a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$ such that

$$\lim_{n\to\infty} \||\xi_n\rangle\!\rangle^{\mathrm{e}} - T\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})} = 0.$$

It follows that

$$\lim_{n\to\infty} \||\xi_n\rangle\!\rangle^{\mathrm{e}}[\![K]\!] - T([\![K]\!])\|_{\mathcal{E}} = 0.$$

Next, for $n \in \mathbb{N}$, we have $|\xi_n\rangle\!\rangle^{\mathrm{e}}[\![K]\!] = \xi_n * [\![K^{\vee}]\!] \subseteq \mathcal{R} * L^{\infty,c}(G,B) \subseteq \mathcal{R} \subseteq \mathcal{E}_{\mathrm{si}}$, so for all $m,n \in \mathbb{N}$,

$$\begin{aligned}
&\||\xi_m\rangle\!\rangle^{\mathrm{e}}[\![K]\!] - |\xi_n\rangle\!\rangle^{\mathrm{e}}[\![K]\!]\|_{\mathcal{E}_{\mathrm{si}}} \\
&= \|(|\xi_m\rangle\!\rangle^{\mathrm{e}} - |\xi_n\rangle\!\rangle^{\mathrm{e}})[\![K]\!]\|_{\mathcal{E}_{\mathrm{si}}} \\
&= \|(|\xi_m\rangle\!\rangle^{\mathrm{e}} - |\xi_n\rangle\!\rangle^{\mathrm{e}})[\![K]\!]\|_{\mathcal{E}} + \||(|\xi_m\rangle\!\rangle^{\mathrm{e}} - |\xi_n\rangle\!\rangle^{\mathrm{e}})[\![K]\!]\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})} \\
&= \|(|\xi_m\rangle\!\rangle^{\mathrm{e}} - |\xi_n\rangle\!\rangle^{\mathrm{e}})[\![K]\!]\|_{\mathcal{E}} + \|(|\xi_m\rangle\!\rangle^{\mathrm{e}} - |\xi_n\rangle\!\rangle^{\mathrm{e}}) \circ |[\![K]\!]\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})} \\
&\leq \||\xi_m\rangle\!\rangle^{\mathrm{e}} - |\xi_n\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})} \cdot \|[\![K]\!]\|_{\mathcal{L}} + \||\xi_m\rangle\!\rangle^{\mathrm{e}} - |\xi_n\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})} \cdot \||[\![K]\!]\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(\mathcal{L})} \\
&= \||\xi_m\rangle\!\rangle^{\mathrm{e}} - |\xi_n\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})} \cdot \|[\![K]\!]\|_{\mathcal{L}_{\mathrm{si}}}.
\end{aligned}$$

As $(|\xi_n\rangle\!\rangle^{\mathrm{e}})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\left(\mathcal{F}, \|\cdot\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})}\right)$, it follows that $(|\xi_n\rangle\!\rangle^{\mathrm{e}}[\![K]\!])$ is a Cauchy sequence in $\left(\mathcal{R}, \|\cdot\|_{\mathcal{E}_{\mathrm{si}}}\right)$. However, $\left(\mathcal{R}, \|\cdot\|_{\mathcal{E}_{\mathrm{si}}}\right)$ is a Banach space, so there exists an $\eta \in \mathcal{R}$ such that

$$\lim_{n\to\infty} \||\xi_n\rangle\!\rangle^{\mathrm{e}}[\![K]\!] - \eta\|_{\mathcal{E}_{\mathrm{si}}} = 0.$$

It is easily seen from the definition of $\|\cdot\|_{\mathcal{E}_{\mathrm{si}}}$ that convergence with respect to $\|\cdot\|_{\mathcal{E}_{\mathrm{si}}}$ implies convergence with respect to $\|\cdot\|_{\mathcal{E}}$. Hence, we in fact have

$$\lim_{n\to\infty} \||\xi_n\rangle\!\rangle^{\mathrm{e}}[\![K]\!] - \eta\|_{\mathcal{E}} = 0.$$

Therefore, $T([\![K]\!]) = \eta \in \mathcal{R}$, which shows that $\mathcal{F}[L^{\infty,c}(G,B)] \subseteq \mathcal{R}$.

From what we have proven so far, we have

$$x[\mathcal{F}[L^{\infty,c}(G,B)]] = (x \circ \mathcal{F})[L^{\infty,c}(G,B)] \subseteq \mathcal{F}[L^{\infty,c}(G,B)] \subseteq \mathcal{R}.$$

We next to show that $\mathcal{F}[L^{\infty,c}(G,B)]$ is dense in $\mathcal{R}$ with respect to $\|\cdot\|_{\mathcal{E}_{\mathrm{si}}}$.

Under Condition (S), we have that $\mathcal{R} * L^{\infty,c}(G,B) = |\mathcal{R}\rangle\!\rangle^{\mathrm{e}}[L^{\infty,c}(G,B)]$ is dense in $\mathcal{R}$ with respect to $\|\cdot\|_{\mathcal{E}_{\mathrm{si}}}$. However,

$$|\mathcal{R}\rangle\!\rangle^{\mathrm{e}}[L^{\infty,c}(G,B)] \subseteq \mathcal{F}[L^{\infty,c}(G,B)] \subseteq \mathcal{R},$$

and so $\mathcal{F}[L^{\infty,c}(G,B)]$ is indeed dense in $\mathcal{R}$ with respect to $\|\cdot\|_{\mathcal{E}_{\mathrm{si}}}$.

Finally, we will show that $x[\mathcal{R}] \subseteq \mathcal{R}$. Let $\zeta \in \mathcal{R}$, and choose a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $\mathcal{F}[L^{\infty,c}(G,B)]$ such that

$$\lim_{n\to\infty} \|\zeta_n - \zeta\|_{\mathcal{E}_{\mathrm{si}}} = 0.$$

As mentioned earlier, convergence with respect to $\|\cdot\|_{\mathcal{E}_{\mathrm{si}}}$ implies convergence with respect to $\|\cdot\|_{\mathcal{E}}$, so the limit above implies that

$$\lim_{n\to\infty} \|x(\zeta_n) - x(\zeta)\|_{\mathcal{E}} = 0.$$

Furthermore, for all $m,n \in \mathbb{N}$, we have

$$\begin{aligned}
\|x(\zeta_m) - x(\zeta_n)\|_{\mathcal{E}_{\mathrm{si}}} &= \|x(\zeta_m - \zeta_n)\|_{\mathcal{E}_{\mathrm{si}}} \\
&= \|x(\zeta_m - \zeta_n)\|_{\mathcal{E}} + \||x(\zeta_m - \zeta_n)\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})} \\
&= \|x(\zeta_m - \zeta_n)\|_{\mathcal{E}} + \|x \circ |\zeta_m - \zeta_n\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})} \\
&\leq \|x\|_{\mathbb{L}^G(\mathcal{E})} \cdot \|\zeta_m - \zeta_n\|_{\mathcal{E}} + \|x\|_{\mathbb{L}^G(\mathcal{E})} \cdot \||\zeta_m - \zeta_n\rangle\!\rangle^{\mathrm{e}}\|_{\mathbb{L}^G(\mathcal{L},\mathcal{E})}
\end{aligned}$$

$$= \|x\|_{\mathbb{L}^G(\mathcal{E})} \cdot \|\zeta_m - \zeta_n\|_{\mathcal{E}_{\mathrm{si}}}.$$

As $(\zeta_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\left(\mathcal{R}, \|\cdot\|_{\mathcal{E}_{\mathrm{si}}}\right)$, it follows that $(x(\zeta_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\left(\mathcal{R}, \|\cdot\|_{\mathcal{E}_{\mathrm{si}}}\right)$ also. By virtue of $\left(\mathcal{R}, \|\cdot\|_{\mathcal{E}_{\mathrm{si}}}\right)$ being a Banach space, there exists an $\eta \in \mathcal{R}$ such that

$$\lim_{n \to \infty} \|x(\zeta_n) - \eta\|_{\mathcal{E}_{\mathrm{si}}} = 0.$$

By now, it should be easily noticed that

$$\lim_{n \to \infty} \|x(\zeta_n) - \eta\|_{\mathcal{E}} = 0.$$

Hence, $x(\zeta) = \eta \in \mathcal{R}$, which shows that $x[\mathcal{R}] \subseteq \mathcal{R}$. Similarly, $x^*[\mathcal{R}] \subseteq \mathcal{R}$. Therefore, x is an $(\mathcal{R}, \mathcal{R})$-continuous operator. $\qquad \square$

We conclude that the range of ϕ is precisely the space of $(\mathcal{R}, \mathcal{R})$-continuous operators. $\qquad \square$

From Theorem 7, we thus see that if the C^*-dynamical system (G, B, β, ω) satisfies Condition (S), then we have an equivalence between isomorphism classes of continuously square-integrable (G, B, β, ω)-modules and isomorphism classes of Hilbert (G, B, β, ω)-modules. As of the moment, it is not clear how Condition (S) can be relaxed or if it can be removed altogether without affecting the main result.

11. Appendix

In what follows, (X, Σ, μ) is σ-finite measure space and B a separable Banach space.

Recall that a function $f : X \to B$ is by definition Bochner-measurable iff it is the pointwise limit of a sequence of integrable simple functions from X to B. The next lemma shows that we may replace 'integrable simple functions' by just 'simple functions' at no cost, thus simplifying the definition.

Lemma 6. *The function $f : X \to B$ is Bochner-measurable iff it is the pointwise limit of a sequence of simple functions from X to B.*

Proof. If $f : X \to B$ is Bochner-measurable, then it is by definition already the pointwise limit of a sequence of simple functions from X to B.

Conversely, suppose that $f : X \to B$ is the pointwise limit of a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of simple functions from X to B. As (X, Σ, μ) is σ-finite, we can find a sequence $(E_n)_{n \in \mathbb{N}}$ in Σ^f such that

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots \quad \text{and} \quad X = \bigcup_{n=1}^{\infty} E_n.$$

Then $(\chi_{E_n} \cdot \sigma_n)_{n \in \mathbb{N}}$ is a sequence of integrable simple functions from X to B converging pointwise to f, so f is Bochner-measurable. $\qquad \square$

Lemma 7. *A function $f : X \to B$ is Bochner-measurable iff it is Borel-measurable.*

Proof. If $f : X \to B$ is Borel-measurable, then by Proposition E.2 of [3], it is the pointwise limit of a sequence of simple functions from X to B. By the previous lemma, f is Bochner-measurable.

If $f : X \to B$ is Bochner-measurable, then f is Borel-measurable by Proposition E.1 of [3]. $\qquad \square$

As our two notions of measurability — Bochner-measurability and Borel-measurability — coincide, we will no longer distinguish them and simply call a function that is measurable in either sense 'measurable'.

Lemma 8. *If $f : G \to B$ is the pointwise limit of a sequence of measurable functions from G to B, then f is measurable.*

Proof. This is immediate from Proposition E.1 of [3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 8. *The set of measurable functions $f : X \to B$ is closed under pointwise addition, pointwise scalar multiplication and pointwise multiplication, i.e., it is an algebra under pointwise operations. If B is further assumed to be a C^*-algebra, then this set is closed under pointwise involution as well, thus making it a $*$-algebra.*

Proof. The proof is straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, let G be a second-countable, locally compact Hausdorff group. Denote its Borel σ-algebra by Σ_g, and let μ denote a Haar measure on G. Then the measure space (G, Σ_G, μ) is σ-compact, hence σ-finite. As mentioned earlier on in this paper, there will be a need to complete (G, Σ_G, μ), and we will denote the completion by the same triple.

We furthermore strengthen our assumption on B to that of a separable C^*-algebra.

Theorem 9. *The function $\begin{Bmatrix} G & \to & B \\ x & \mapsto & \phi_1(x)^*\phi_2(x) \end{Bmatrix}$ is Bochner-integrable for all $\phi_1, \phi_2 \in \mathscr{L}^2(G, B)$.*

Proof. The function
$$\begin{Bmatrix} G & \to & B \\ x & \mapsto & \phi_1(x)^*\phi_2(x) \end{Bmatrix}$$
is measurable, as pointwise involution and pointwise multiplication respect measurability. Hence,
$$\begin{Bmatrix} G & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \left\|\phi_1(x)^*\phi_2(x)\right\|_B \end{Bmatrix}$$
is measurable. Both
$$\begin{Bmatrix} G & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \|\phi_1(x)\|_B^2 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} G & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \|\phi_2(x)\|_B^2 \end{Bmatrix}$$
are integrable by hypothesis, so by Hölder's Inequality,
$$\begin{Bmatrix} G & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \|\phi_1(x)\|_B \|\phi_2(x)\|_B \end{Bmatrix}$$
is integrable. Then as
$$\left\|\phi_1(x)^*\phi_2(x)\right\|_B \leq \|\phi_1(x)\|_B \|\phi_2(x)\|_B$$
for all $x \in G$, it follows that
$$\begin{Bmatrix} G & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \left\|\phi_1(x)^*\phi_2(x)\right\|_B \end{Bmatrix}$$
is also integrable. Therefore,
$$\begin{Bmatrix} G & \to & B \\ x & \mapsto & \phi_1(x)^*\phi_2(x) \end{Bmatrix}$$
is Bochner-integrable by Bochner's Integrability Criterion. $\qquad\qquad\qquad\qquad\square$

Theorem 10. *The right B-action $\bullet$ on $L^2(G, B)$ is non-degenerate, i.e, $\overline{L^2(G, B) \bullet B}^{\|\cdot\|_2} = L^2(G, B)$.*

Proof. Let $\phi \in \mathscr{L}^2(G, B)$ and $(e_n)_{n \in \mathbb{N}}$ a countable approximate identity for B. It is inherent from the definition of an approximate identity that $e_n \in \mathbb{B}(B_\geq)$ for all $n \in \mathbb{N}$. Hence,

$$\|\phi(x)e_n\|_B \leq \|\phi(x)\|_B$$

for all $x \in G$ and $n \in \mathbb{N}$, so $[\![\phi]\!] \bullet e_n \in L^2(G, B)$ for all $n \in \mathbb{N}$. Furthermore,

$$\lim_{n \to \infty} \phi(x)e_n = \phi(x)$$

for all $x \in G$. Consequently, the sequence of functions

$$\left(\left\{ \begin{array}{ccc} G & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \|\phi(x) - \phi(x)e_n\|_B^2 \end{array} \right\} \right)_{n \in \mathbb{N}}$$

converges pointwise to the zero function. As this sequence is dominated by

$$\left\{ \begin{array}{ccc} G & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & 4\|\phi(x)\|_B^2 \end{array} \right\} \in \mathscr{L}^1(G),$$

Lebesgue's Dominated Convergence Theorem says that $([\![\phi]\!] \bullet e_n)_{n \in \mathbb{N}}$ converges to $[\![\phi]\!]$. Therefore,

$$[\![\phi]\!] \in \overline{L^2(G, B) \bullet B}^{\|\cdot\|_2},$$

and as ϕ is arbitrary, we are done. $\qquad\square$

Lemma 9. *If $F : G \times G \to B$ is measurable, then $F(x, \cdot) : G \to B$ is measurable for each $x \in G$. Similarly, $F(\cdot, y) : G \to B$ is measurable for each $y \in G$.*

Proof. As F is measurable, there is a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of simple functions from $G \times G$ to B converging to F pointwise. Let $(\mathcal{S}_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of $\Sigma_{G \times G} \times B$ such that

$$\sigma_n = \sum_{(E, b) \in \mathcal{S}_n} \chi_E \cdot b$$

for each $n \in \mathbb{N}$. Fixing $x \in G$, we then have

$$F(x, y) = \lim_{n \to \infty} \sigma_n(x, y) = \lim_{n \to \infty} \sum_{(E, b) \in \mathcal{S}_n} \chi_E(x, y) \cdot b = \lim_{n \to \infty} \sum_{(E, b) \in \mathcal{S}_n} \chi_{E_x}(y) \cdot b$$

for all $y \in G$. Define a sequence $(\tau_n^x)_{n \in \mathbb{N}}$ of simple functions from G to B by

$$\forall n \in \mathbb{N} : \quad \tau_n^x := \sum_{(E, b) \in \mathcal{S}_n} \chi_{E_x} \cdot b.$$

Then $(\tau_n^x)_{n \in \mathbb{N}}$ converges pointwise to $F(x, \cdot)$, so $F(x, \cdot)$ is measurable.

The proof that $F(\cdot, y)$ is measurable for each $y \in G$ is similar and is omitted. $\qquad\square$

Theorem 11. $F_\phi := \left\{ \begin{array}{ccc} G \times G & \to & B \\ (s, x) & \mapsto & \omega\big(s, s^{-1}x\big)^* \beta_s\big(\phi\big(s^{-1}x\big)\big) \end{array} \right\}$ *is measurable for each $\phi \in \mathscr{L}^2(G, B)$.*

Proof. Let $\phi \in \mathscr{L}^2(G, B)$. Define a homeo-automorphism $h : G \times G \to G \times G$ by

$$\forall (s, x) \in G \times G : \quad h(s, x) := (s, sx).$$

It clearly suffices to prove that $F_\phi \circ h : G \times G \to B$ is measurable.

As ϕ is measurable, there is a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of simple functions from G to B converging pointwise to ϕ. Let $(\mathcal{S}_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of $\Sigma_G \times B$ such that

$$\sigma_n = \sum_{(S, b) \in \mathcal{S}_n} \chi_S \cdot b$$

for each $n \in \mathbb{N}$. Then

$$(F_\phi \circ h)(s, x) = \omega(s, x)^* \beta_s(\phi(x)) = \lim_{n \to \infty} \sum_{(S, b) \in \mathcal{S}_n} \chi_S(s, x) \omega(s, x)^* \beta_s(b)$$

for all $(s, x) \in G \times G$. Once we show that

$$\left\{ \begin{matrix} G \times G & \to & B \\ (s, x) & \mapsto & \chi_S(s, x) \omega(s, x)^* \beta_s(b) \end{matrix} \right\}$$

is measurable for each $n \in \mathbb{N}$ and $(S, b) \in \mathcal{S}_n$, we are done. In turn, we only need to establish that

$$H_b := \left\{ \begin{matrix} G \times G & \to & B \\ (s, x) & \mapsto & \omega(s, x)^* \beta_s(b) \end{matrix} \right\}$$

is measurable for each $b \in B$, thanks to the continuity of scalar multiplication.

Now, fix $b \in B$. By the definition of a twisted C^*-dynamical system,

$$f_b := \left\{ \begin{matrix} G & \to & B \\ s & \mapsto & \beta_s(b) \end{matrix} \right\}$$

is measurable, so there is a sequence $(\tau_n)_{n \in \mathbb{N}}$ of simple functions from G to B converging pointwise to f_b. Let $(\mathcal{T}_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of $\Sigma_G \times B$ such that

$$\tau_n = \sum_{(T, b') \in \mathcal{T}_n} \chi_T \cdot b'$$

for each $n \in \mathbb{N}$. Then

$$H_b(s, x) = \lim_{n \to \infty} \sum_{(T, b') \in \mathcal{T}_n} \chi_T(s) \omega(s, x)^* b'$$

for all $(s, x) \in G \times G$. Therefore, H_b is the pointwise limit of a sequence of measurable functions from $G \times G$ to B, so H_b is measurable by Lemma 8. As b is arbitrary, we are done. $\square$

Theorem 12. *We have $\Gamma_s(\llbracket \phi \rrbracket) \in L^2(G, B)$ for each $s \in G$ and each $\phi \in \mathscr{L}^2(G, B)$.*

Proof. Fix $\phi \in \mathscr{L}^2(G, B)$. We have already seen that

$$\left\{ \begin{matrix} G \times G & \to & B \\ (s, x) & \mapsto & \omega(s, s^{-1}x)^* \beta_s(\phi(s^{-1}x)) \end{matrix} \right\}$$

is measurable. Fixing an $s \in G$, we see from Lemma 9 that $\Gamma_s(\llbracket \phi \rrbracket) : G \to B$ is measurable.

It remains to prove that $\Gamma_s(\phi)$ is square-integrable. To begin, observe that

$$\begin{aligned} \forall x \in G : \quad & \beta_s\big(\phi\big(s^{-1}x\big)\big)^* \omega\big(s, s^{-1}x\big) \omega\big(s, s^{-1}x\big)^* \beta_s\big(\phi\big(s^{-1}x\big)\big) \\ &= \beta_s\big(\phi\big(s^{-1}x\big)\big)^* \beta_s\big(\phi\big(s^{-1}x\big)\big) \\ &= \beta_s\big(\phi\big(s^{-1}x\big)^*\big) \beta_s\big(\phi\big(s^{-1}x\big)\big) \\ &= \beta_s\big(\phi\big(s^{-1}x\big)^* \phi\big(s^{-1}x\big)\big). \end{aligned}$$

Hence,

$$\begin{aligned} \int_G \|\Gamma_s(\llbracket \phi \rrbracket)\|_B^2 \, \mathrm{d}x &= \int_G \big\| \Gamma_s(\llbracket \phi \rrbracket)^* \, \Gamma_s(\llbracket \phi \rrbracket) \big\|_B \, \mathrm{d}\mu \\ &= \int_G \Big\| \beta_s\Big(\phi\big(s^{-1}x\big)^* \phi\big(s^{-1}x\big)\Big) \Big\|_B \, \mathrm{d}x \\ &= \int_G \Big\| \phi\big(s^{-1}x\big)^* \phi\big(s^{-1}x\big) \Big\|_B \, \mathrm{d}x \end{aligned}$$

$$= \int_G \left\| \phi(x)^* \phi(x) \right\|_B \, \mathrm{d}x$$

$$= \int_G \left\| \phi(x) \right\|_B^2 \, \mathrm{d}x$$

$$< \infty,$$

which proves that $\Gamma_s(\llbracket \phi \rrbracket) \in L^2(G, B)$. $\qquad\square$

Theorem 13. *The Hilbert-module operator $\Gamma_s : L^2(G, B) \to L^2(G, B)$ is isometric and adjointable for each $s \in G$.*

Proof. Fix $s \in G$. From the proof of the previous theorem, it is clear that Γ_s is an isometry. A formula for the adjoint can be easily found via a standard argument. $\qquad\square$

Theorem 14. *If $F : G \times G \to B$ is measurable and $F(x, \cdot) \in \mathscr{L}^2(G, B)$ for each $x \in G$, then*

$$\begin{Bmatrix} G & \to & L^2(G, B) \\ x & \mapsto & \llbracket F(x, \cdot) \rrbracket \end{Bmatrix}$$

is measurable.

Proof. Let $(P_n)_{n \in \mathbb{N}}$ be a partition of G into disjoint measurable subsets of finite measure, and define a sequence $(P_{m,n,k})_{(m,n,k) \in \mathbb{N}^2 \times \mathbb{N}_0}$ by

$$\forall (m, n, k) \in \mathbb{N}^2 \times \mathbb{N}_0 : \quad P_{m,n,k} := \{ (x, y) \in P_{m,n} \mid k \leq \| F(x, y)) \|_B < k + 1 \}.$$

Next, define

$$(F_{m,n,k} : G \times G \to B)_{(m,n,k) \in \mathbb{N}^2 \times \mathbb{N}_0} := \left(\chi_{P_{m,n,k}} \cdot F \right)_{(m,n,k) \in \mathbb{N}^2 \times \mathbb{N}_0}.$$

Clearly,

$$\forall (x, y) \in G \times G : \quad x(y) = \lim_{N \to \infty} \sum_{m,n,k=1}^{N} F_{m,n,k}(x, y).$$

Let $x \in G$. As

$$\left(\begin{Bmatrix} G & \to & \mathbb{R}_{\geq 0} \\ y & \mapsto & \left\| \sum_{m,n,k=1}^{N} F_{m,n,k}(x, y) \right\|_B^2 \end{Bmatrix} \right)_{N \in \mathbb{N}} \quad \text{is dominated by} \quad \begin{Bmatrix} G & \to & \mathbb{R}_{\geq 0} \\ y & \mapsto & \| F(x, y) \|_B^2 \end{Bmatrix},$$

it follows from Lebesgue's Dominated Convergence Theorem that

$$\left(\sum_{m,n,k=1}^{N} F_{m,n,k}(x, \cdot) : G \to B \right)_{N \in \mathbb{N}} \quad \text{converges to } F(x, \cdot) \text{ in } \mathscr{L}^2(G, B).$$

In other words,

$$\left(\begin{Bmatrix} G & \to & \mathscr{L}^2(G, B) \\ x & \mapsto & \sum_{m,n,k=1}^{N} F_{m,n,k}(x, \cdot) \end{Bmatrix} \right)_{N \in \mathbb{N}} \quad \text{converges pointwise to} \quad \begin{Bmatrix} G & \to & \mathscr{L}^2(G, B) \\ x & \mapsto & F(x, \cdot) \end{Bmatrix}.$$

By Lemma 8, it suffices to prove that

$$\begin{Bmatrix} G & \to & L^2(G, B) \\ x & \mapsto & \llbracket F_{m,n,k}(x, \cdot) \rrbracket \end{Bmatrix}$$

is measurable for each $(m, n, k) \in \mathbb{N}^2 \times \mathbb{N}_0$.

Fix $(m, n, k) \in \mathbb{N}^2 \times \mathbb{N}_0$. Let $(\sigma_i : G \times G \to B)_{i \in \mathbb{N}}$ be a sequence of simple functions, with support in $P_m \times P_n$, that is dominated in the sup norm by $k + 1$ and that converges to $F_{m,n,k}$ pointwise. The existence of such a sequence is made possible by the separability of B and the measurability of $F_{m,n,k}$.

Let $x \in G$. As

$$\left(\left\{ \begin{matrix} G & \to & \mathbb{R}_{\geq 0} \\ y & \mapsto & \|\sigma_i(x, y)\|_B^2 \end{matrix} \right\} \right)_{i \in \mathbb{N}} \text{ is dominated by } (k + 1)^2 \cdot \chi_{P_n}$$

and converges to

$$\left\{ \begin{matrix} G & \to & \mathbb{R}_{\geq 0} \\ y & \mapsto & \|F_{m,n,k}(x, y)\|_B^2 \end{matrix} \right\}$$

pointwise, it follows from Lebesgue's Dominated Convergence Theorem that $(\sigma_i(x, \cdot))_{i \in \mathbb{N}}$ converges to $F_{m,n,k}(x, \cdot)$ in $\mathscr{L}^2(G, B)$. In other words,

$$\left(\left\{ \begin{matrix} G & \to & \mathscr{L}^2(G, B) \\ x & \mapsto & \sigma_i(x, \cdot) \end{matrix} \right\} \right)_{i \in \mathbb{N}} \text{ converges pointwise to } \left\{ \begin{matrix} G & \to & \mathscr{L}^2(G, B) \\ x & \mapsto & F_{m,n,k}(x, \cdot) \end{matrix} \right\}.$$

By Lemma 8 again, it suffices to prove that

$$\left\{ \begin{matrix} G & \to & L^2(G, B) \\ x & \mapsto & [\![\sigma_i(x, \cdot)]\!] \end{matrix} \right\}$$

is measurable for each $i \in \mathbb{N}$.

Finally, suppose that E is a measurable subset of $P_m \times P_n$. By the Carathéodory construction of the Haar measure on $G \times G$, there exists a sequence $(R_j)_{j \in \mathbb{N}}$ of finite collections of disjoint measurable rectangles, all contained in $P_m \times P_n$, with sides of finite measure, such that $\left(\sum_{S \times T \in R_j} \chi_{S \times T} \right)_{j \in \mathbb{N}}$ converges to χ_E pointwise almost-everywhere. As

$$\left\{ \begin{matrix} G & \to & L^2(G, B) \\ x & \mapsto & [\![\chi_{S \times T}(x, \cdot)]\!] \end{matrix} \right\} = \left\{ \begin{matrix} G & \to & L^2(G, B) \\ x & \mapsto & \chi_S(x) \cdot [\![\chi_T]\!] \end{matrix} \right\}$$

is measurable (being a simple $L^2(G, B)$-valued function on G), we see that

$$\left\{ \begin{matrix} G & \to & L^2(G, B) \\ x & \mapsto & [\![\chi_E(x, \cdot)]\!] \end{matrix} \right\}$$

is measurable. This, then, concludes the proof. $\qquad\square$

Corollary 1. *The mapping*

$$\left\{ \begin{matrix} G & \to & L^2(G, B) \\ s & \mapsto & \Gamma_s([\![\phi]\!]) \end{matrix} \right\}$$

is measurable for each $\phi \in \mathscr{L}^2(G, B)$, i.e.,

$$\left\{ \begin{matrix} G & \to & \mathrm{Isom}\big(L^2(G, B)\big) \\ s & \mapsto & \Gamma_s \end{matrix} \right\}$$

is strongly measurable.

The results in this section can be generalized to the case where B is not separable. This is because by the definition of measurability, off a null subset of the domain measure space, we can simply redefine a measurable function to attain the value 0 on that set so as to obtain a function that is separably valued. The values of integrals are not affected by such a redefinition.

References

[1] J. Brown, *Proper Actions of Groupoids on C^*-Algebras (PhD thesis)* (2009).

[2] R. Busby and H. Smith, *Representations of Twisted Group Algebras*, Trans. Amer. Math. Soc. **149** (1970), 503–537. MR 0264418.

[3] D. Cohn, *Measure Theory*, Birkhäuser, Boston, MA, 1980. MR 0578344.

[4] R. Exel, *Morita-Rieffel Equivalence and Spectral Theory for Integrable Automorphism Groups of C^*-Algebras*, J. Funct. Anal. **172** (2000), no. 2, 404–465. MR 1753180.

[5] R. Meyer, *Generalized Fixed Point Algebras and Square-Integrable Group Representations*, J. Funct. Anal. **186** (2001), no. 1, 167–195. MR 1863296.

[6] J. Packer and I. Raeburn, *Twisted Crossed Products of C^*-Algebras*, Math. Proc. Cambridge Philos. Soc. **106** (1989), no. 2, 293–311. MR 1002543.

[7] W. Paschke, *Inner Product Modules over B^*-Algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468. MR 0355613.

[8] M. Rieffel, *Proper Actions of Groups on C^*-Algebras*, Mappings of Operator Algebras **84** (1990), 141–182. MR 1103376.

[9] _____, *Integrable and Proper Actions on C^*-Algebras, and Square-integrable Representations of Groups*, Expo. Math. **22** (2004), no. 1, 1–53. MR 2166968.

[10] D. Williams, *Crossed Products of C^*-Algebras*, Math. Surveys Monogr., vol. 134, American Mathematical Society, Providence, RI, 2007. MR 2288954.

Department of Mathematics, The University of Kansas, 1460 Jayhawk Blvd, Lawrence, KS 66045, USA

E-mail address: lhuang@ku.edu